**HOMEWORK 4 - Math 2921 - ODE II** RUBIN - Winter/Spring '09

This assignment is due in class on Friday, March 27th, 2009. It involves a lot of calculations!

---

1. Fill in the details from Carr, section 1.4, Example 1, and complete Exercise 1. That is, for the system

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x, \\ \dot{y} &= -y + cx^2 + dx^2y, \end{aligned}$$

with center manifold $y = h(x)$, approximate $h(x)$ and write an equation for the flow on the center manifold as follows:

a) approximate $h(x)$ up to $O(x^2)$ and plug into the $x$-equation in the system (Carr replaces $x$ by $u$ when he does this, but there is no reason for you to do so);

b) if $a + c = 0$, then approximate $h(x)$ up to $O(x^4)$ and use this in the $x$-equation;

c) <u>bonus</u>: if $a + c = cd + bc^2 = 0$, then approximate $h(x)$ up to $O(x^6)$ and use this in the $x$-equation.

2. Consider the system

$$\begin{aligned} \dot{x} &= -x - y + z^2, \\ \dot{y} &= 2x + y - z^2, \\ \dot{z} &= x + 2y - z. \end{aligned}$$

a) Show that the system has a critical point with a 2-d center manifold.

b) Change coordinates to $(u, v, w)$ such that $u = h(v, w)$ on the center manifold.

c) Characterize the shape of the center manifold by finding the leading terms of $h(v, w)$.

d) Generate a phase portrait representing the $(v, w)$ dynamics on the center manifold (possibly using software).

e) Translate your results to the original coordinates.

3. Use center manifold techniques to sketch the bifurcation diagram and phase portrait in the vicinity of the origin for

$$\begin{aligned} \dot{u} &= -u - \epsilon v - v^3, \\ \dot{v} &= u, \end{aligned}$$

where $\epsilon$ is a small parameter. <u>NOTES</u>:

a) When you transform your system to appropriate coordinates, you can use the $\epsilon = 0$ eigenvectors.

b) When you sketch your phase portrait, please give an example picture for $\epsilon < 0$ and another for $\epsilon > 0$. Feel free to label your sketch liberally. Be sure to include a few representative trajectories off of the invariant manifolds, as well as the invariant manifolds themselves. You should use the center manifold analysis to make these pictures precise, to leading order.

4. Guckenheimer and Holmes, pg. 153, exercise 3.4.7: Show that the system $\ddot{x} + \mu\dot{x} + \nu x + x^2\dot{x} + x^3 = 0$ undergoes Andronov-Hopf bifurcations on the lines $B_1 = \{\mu = 0 | \nu > 0\}$ and $B_2 = \{\mu = \nu | \mu, \nu < 0\}$. Show that the former is supercritical and occurs at the fixed point (0,0) while the latter is subcritical and occurs simultaneously at $(x, \dot{x}) = (\pm\sqrt{-\nu}, 0)$. Attempt to sketch the phase portraits of this system for various representative values of $(\mu, \nu) \in \mathbb{R}^2$. What is the form of the degenerate singularity occurring at $(x, \dot{x}) = (0, 0)$ when $(\mu, \nu) = (0, 0)$?

5. Demonstrate that a Andronov-Hopf bifurcation occurs in the Brusselator model

$$\begin{aligned} \dot{x} &= c - (d+1)x + x^2 y \\ \dot{y} &= dx - x^2 y \end{aligned}$$

with $c, d > 0$ as $d$ varies (with $c$ fixed). Further, compute the normal form for this bifurcation up to third order (as done in class). To do this, first define new variables, say $(\xi, \eta)$, that give a translation such that the relevant critical point is at $(0, 0)$. Use eigenvectors to transform from $(\xi, \eta)$ to new variables, say $(u, v)$, for which the differential equations at the bifurcation point take the form $\dot{u} = c_1 v + f(u, v)$, $\dot{v} = c_2 u + g(u, v)$ for appropriate constants $c_1, c_2$. Finally, use this last system to compute the normal form and check whether this bifurcation is subcritical or supercritical. Sketch the generic bifurcation diagram for the type that you find, making sure to label which structures are stable and unstable.