HOMEWORK 3 - Math 2921 - ODE II RUBIN - Winter/Spring '09

This assignment is due in class on Monday, March 2nd, 2009.

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1. Note that in our example of the Lyapunov-Perron approach to existence of a stable manifold, we obtained
$$y(t; x_0, h(x_0)) = h(x_0)e^{-t} + e^{-t} \int_0^t e^{\tau} x^2(\tau; x_0, h(x_0)) d\tau.$$
In general, an analogous (more general) expression can be obtained from the variation of parameters formula, using projections onto stable and unstable components (assuming no center directions are present). For $\dot{x} = Ax + G(x)$, where $G(x)$ is higher order and $A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ with $Re(\lambda) < 0$ for the eigenvalues of P and $Re(\lambda) > 0$ for the eigenvalues of Q, this yields (Perko, equation (7) of Section 2.7)
$$u(t, a) = U(t)a + \int_0^t U(t-s)G(u(s, a))ds - \int_t^{\infty} V(t-s)G(u(s, a))ds$$
where $x = a$ is an initial condition and
$$U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix}, \; V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}. \tag{1}$$
Equation (1) can be used as an iteration procedure to derive stable (and unstable, by time reversal and exchange of components) manifolds; see e.g. Example 2 from Perko Section 2.7. Use this to complete problem 4 from Perko Section 2.7 (pg. 114 in my copy).

2. Prove that, under the assumptions of the stable manifold theorem, $W^s_{loc} \subset M := \{x : x \cdot t \in K \cap N, \; \forall \; t \geq 0\}$. HINT: A calculation of the rate of change of a certain quantity, along the lines of what we did in the proof of the stable manifold theorem in class, may be helpful.

3. Consider
$$\begin{aligned} \dot{u} &= \lambda_- u + G_1(u, v), \\ \dot{v} &= \lambda_+ v + G_2(u, v) \end{aligned}$$
where $\lambda_- < 0 < \lambda_+$ and $|G_i(u, v)| \leq \gamma(|u| + |v|)(|u| + |v|), \gamma(x) \to 0$ as $x \to 0$. Prove that $T_0 W^s_{loc}(0) = E_-$. That is, prove that the tangent space to the local stable manifold of the origin is the stable subspace derived by linearizing the system of ODE at the origin. NOTE: You do not need to show all details if you wish to use modified versions of certain steps of the stable manifold theorem proof for this problem, as long as you are precise about the modifications and implications.

4. Recall that in class, we proved the existence of a traveling wave solution $u(z) = u(x - ct)$ of
$$u_t = u_{xx} + u(1 - u) \tag{2}$$
for $c > 0$. Prove that for $c \geq 2$, this wave is positive; that is, $u(z) > 0$ for all z.

5. Show that for $c \geq 2$, there exists a *positive* traveling wave solution $u(z) = u(x - ct)$ of

$$u_t = u_{xx} + u(1-u)^2$$

satisfying

$$\begin{aligned} (u(z), \dot{u}(z)) &\to (1,0) \quad \text{as} \quad z \to -\infty, \\ (u(z), \dot{u}(z)) &\to (0,0) \quad \text{as} \quad z \to \infty. \end{aligned}$$

HINT: There will be a complication here that did not arise for equation (2). The proof from class for (2) can be adjusted accordingly, if you are creative!

6. Give a proof of the center manifold theorem using the implicit function theorem, given the existence of C^1 center-stable and center-unstable manifolds for an ordinary differential equation $\dot{x} = f(x)$ with $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ a sufficiently smooth function and $f(0) = 0$. NOTE: It may be helpful to recall that the center-stable manifold is tangent to $E^- \oplus E^0$ at 0, and the center-unstable manifold is tangent to $E^0 \oplus E^+$ at 0.