HOMEWORK 2 - Math 2921 - ODE II RUBIN - Winter/Spring '09

This assignment is due in class on Monday, February 9th, 2009.

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1. (Chicone, pg. 78, exercise 1.106) Compute the time required for the solution of the system

$$\begin{aligned}\dot{x} &= x(1-y),\\ \dot{y} &= y(x-1),\end{aligned}$$

with initial condition $(x, y) = (1, 0)$, to arrive at the point $(x, y) = (2, 0)$. Note that this system has a section map $y \mapsto h(y)$ defined from a neighborhood of $(x, y) = (1, 0)$ on the line given by $x = 1$ to the line given by $x = 2$. Compute $h'(0)$.

2. Let Γ be a periodic orbit of $\dot{x} = f(x), x \in \mathbb{R}^2$ and let R denote the bounded region in $\mathbb{R}^2$ with boundary Γ. Prove that there exists a critical point of $\dot{x} = f(x)$ in R.

3. Define x to be a *recurrent point* if $x \cdot t_n \to x$ for some sequence $t_n \to \infty$, under the flow of $\dot{x} = f(x)$. Prove that in $\mathbb{R}^2$, any recurrent point is either a critical point or a point on a periodic orbit.

4. Consider the flow generated by a C^1 vector field on $U \subset \mathbb{R}^2$. Prove that if $x, y \in U$, $x \in \omega(y)$, and x lies on a periodic orbit Γ, then $\omega(y) = \{\Gamma\}$.

5. Consider the forced, damped pendulum

$$\ddot{\theta} + \lambda\dot{\theta} + \sin\theta = \mu$$

for $\lambda > 0$ and μ constant. Take your phase space for $(\theta, \dot{\theta})$ to be the cylinder $\mathbf{S^1} \times \mathbf{R}$, formed by identifying $\{\theta = 0\}$ with $\{\theta = 2\pi\}$. Remember, the flow is *on* the surface of the cylinder and cannot go "inside" the cylinder. Show that if $|\mu| > 1$, then this equation has a globally attracting limit cycle (i.e., a *unique* periodic orbit that is the ω-limit set of every point on the cylinder).

6. Consider the equation

$$\ddot{x} = x^3 - \dot{x}(\dot{x} - 2).$$

Show that there is a point $p = (x, \dot{x})$ with $x < 0$ and $\dot{x} > 0$ for which $\omega(p) = \{(0, 0)\}$.