

This assignment is due in class on Monday, January 26th, 2009.

1. Sketch the phase portrait (in \mathbb{R}^3) of

$$\begin{aligned}\dot{x}_1 &= x_1 + x_1^3 \\ \dot{x}_2 &= 2x_1 + x_2 - 2x_3 + x_1x_2 \\ \dot{x}_3 &= 3x_1 + 2x_2 + x_3\end{aligned}$$

2. Consider the planar system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ with a critical point at $(0, 0)$ and a linearization of the form

$$A = \begin{bmatrix} + & - \\ + & - \end{bmatrix}.$$

Note that the x -nullcline is the set of points where $f(x, y) = 0$ and the y -nullcline is the set of points where $g(x, y) = 0$. Show that if the slope of the x -nullcline at the critical point exceeds that of the y -nullcline, then the critical point is a saddle point.

3. On pg. 90, Perko presents the following theorem.

Theorem. Consider the usual IVP, $\dot{x} = f(x), x(0) = x_0$ on an open set $U \subset \mathbb{R}^n$, with $f \in C^1(U)$ and $x_0 \in U$. Denote the maximal interval of existence for this IVP by $J_{x_0} = (\alpha, \beta)$ with $\alpha < 0 < \beta$. If $\beta < \infty$, then given any compact set $K \subset U$, there exists a $t \in (0, \beta)$ such that $x(t) \notin K$.

- a) A corollary follows very simply from this theorem: if $x_1 = \lim_{t \rightarrow \beta^-} x(t)$ exists and $x_1 \in U$, then $\beta = \infty$. Give the proof.
- b) Show that if $x_1 = \lim_{t \rightarrow \beta^-} x(t)$ exists and $x_1 \in U$, then $f(x_1) = 0$, such that x_1 is a critical point of $\dot{x} = f(x)$. (In fact, you can do this at least two different ways, one using the properties of ω -limit sets and one not. I would like a proof that does not directly use properties of ω -limit sets.)
4. Let Γ_{x_0} be a trajectory of $\dot{x} = f(x), x(0) = x_0$ on $U \subset \mathbb{R}^n$ and assume that Γ^+ is contained in a compact set $K \subset U$. Prove that $d(\Phi(t, x_0), \omega(\Gamma_{x_0})) \rightarrow 0$ as $t \rightarrow \infty$.
5. a) Sketch the phase portrait for $\dot{x} = y, \dot{y} = 4x^3 - 8x$.
- b) Prove that the trajectories that meet at the critical points $(x, y) = (\pm\sqrt{2}, 0)$ form differentiable curves (i.e., that dy/dx is finite but nonzero there).
6. a) Find the function V for which $\dot{x} = x^2 - 2xy, \dot{y} = y^2 - x^2$ is a gradient system, and sketch the phase portrait of this system.
- b) Determine a general form for smooth $f(x, y)$ and $g(x, y)$ for which which $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a gradient system and a general procedure for finding $V(x, y)$ when this holds.