Week 9 **Math 1530** *Advanced Calculus 1* - RUBIN - Fall '03

SCHEDULE: Homework from Week 9 is due in class on **Wednesday, October 29**.

- Wednesday, October 22: start Section 4.1
- Friday, October 24: finish Section 4.1/start Section 4.2
- Monday, October 27: finish Section 4.2/Section 4.3

TOPICS:

Section 4.1: Continuous Mappings - It is finally time to focus on functions (on general metric spaces). In this section we define the limit of a function at a point and what it means for a function to be continuous (at a point or on a set). The main result in this section is that there are four equivalent ways to define continuity of a function, one in terms of epsilons and deltas, one in terms of converging sequences, and two that are topological.

Reading Objectives: After reading Section 4.1, students should be able to:

- Determine at what points the limit of a function does not exist.
- Test whether a function such as $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is continuous at a fixed point x_0.
- List four equivalent ways to define continuity.

homework:

1. a) Prove directly that $f(x) = x^3$ is continuous on $(0, 1)$. That is, for fixed $\epsilon > 0$ and $x_0 \in (0, 1)$, find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |x^3 - x_0^3| < \epsilon$.

   b) Repeat the above on $(0, \infty)$.

2. Let A be a subset of a metric space M and let $f : A \to N$ be a continuous function mapping into a metric space N. Show that for all $B \subset A$, $f(cl(B) \cap A) \subset cl(f(B))$.

3. Define $f(x) : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational or } 0 \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ in lowest terms} \end{cases}$$

   Show that $f(x)$ is continuous at the irrationals and discontinuous at nonzero rationals.

BONUS:

1. Show that the converse of problem 2 holds as well (i.e., if $\forall B \subset A$, we have $f(cl(B) \cap A) \subset cl(f(B))$, then f is continuous).

2. Determine where the function

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational or } 0 \\ m \sin \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ in lowest terms} \end{cases}$$

   is continuous and where it is discontinuous (and prove your result!).

Section 4.2: Images of Compact and Connected Sets - This section can be summarized by the statement that continuous functions map connected, path-connected, and compact sets into connected, path-connected, and compact sets, respectively. This follows rather directly from the various ways to define continuous functions.

Reading Objectives: After reading Section 4.2, students should be able to:

- Determine whether particular sets are connected or compact, based on whether they can be defined as images of connected or compact sets under a continuous mapping (e.g. see Example 4.2.3).

homework:

1. Exercise 4.2.3, pg. 184.
2. Exercise 4.2.5, pg. 184. If the answer is no, then provide a counterexample. If the answer is yes, then be sure to prove the result. (Note: Assume that B is nonempty. Otherwise $A \times B = \emptyset$, which is open, even if A is not open!)

Section 4.3: Operations on Continuous Mappings - The composition, sum, product, and quotient of two continuous functions are all continuous (as long as the function in the denominator of the quotient is nonzero). Except for the composition result, these results are proved using the epsilon-delta notion of continuity; that is, for fixed ϵ and x_0 in the domain of the sum/product/quotient (call it F) that we are interested in, we compute a value of δ sufficiently small such that $d(x, x_0) < \delta$ implies that $\rho(F(x), F(x_0)) < \epsilon$. This δ depends on the values $\delta(\epsilon)$ specified by the continuity of f and g.

Reading Objectives: After reading Section 4.3, students should be able to:

- Apply Theorem 4.3.1 and Corollary 4.3.3 to see immediately that functions such as $e^{x^2+\sin x}$ and $x/(x^2+1)$ are continuous on their domains.
- Understand the idea of how $\delta(\epsilon)$ can be selected to prove the results in Corollary 4.3.3.