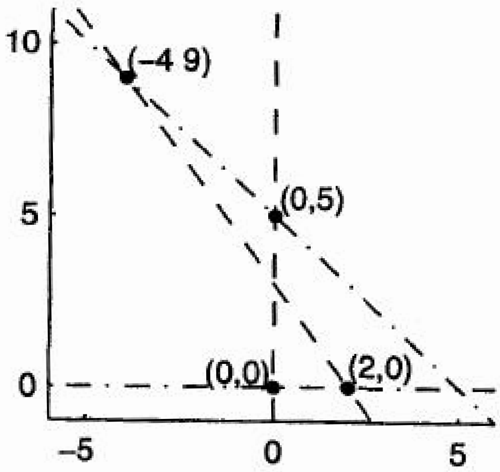
Section 10.1

1. The x-nullcline consists of the two lines defined by $x = 0$ and $3x + 2y = 6$. It is shown dashed in the accompanying figure. The y-nullcline consists of the two lines defined by $y = 0$ and $x + y = 5$. It is shown dot-dashed in the accompanying figure. The equilibrium points are found by looking for the intersections. They are $(0, 0)$, $(0, 5)$, $(2, 0)$, and $(-4, 9)$.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 6 - 6x - 2y & -2x \\ -y & 5 - x - 2y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

has eigenvalues 6 and 5 so $(0, 0)$ is a nodal source. At $(0, 5)$,

$$J(0, 5) = \begin{pmatrix} -4 & 0 \\ -5 & -5 \end{pmatrix}$$

has eigenvalues -4 and -5 so $(0, 5)$ is a nodal sink. At $(2, 0)$,

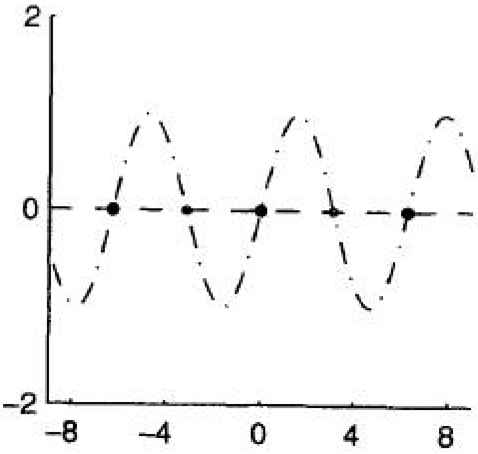
$$J(2, 0) = \begin{pmatrix} -6 & -4 \\ 0 & 3 \end{pmatrix}$$

has eigenvalues -6 and 3 so $(2, 0)$ is a saddle. At $(-4, 9)$,

$$J(-4, 9) = \begin{pmatrix} 12 & 8 \\ -9 & -9 \end{pmatrix}$$

has determinant $D = -36$ so $(-4, 9)$ is a saddle.

7. The x-nullcline consists of the line $y = 0$. It is shown dashed in the accompanying figure. The y-nullcline consists of the curve $y = -\sin x$. It is shown dot-dashed in the accompanying figure. The equilibrium points are where $0 = -\sin x$. Hence $x = k\pi$, where k is any integer. The equilibrium points are $(k\pi, 0)$, where k is any integer.



The Jacobian is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix}.$$

If k is even,

$$J(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

has determinant $D = 1$, trace $T = -1$, and discriminant $T^2 - 4D = -3$, so $(k\pi, 0)$ is a spiral sink. These are shown by solid dots in the figure. If k is odd,

$$J(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

has determinant $D = -1$, so $(k\pi, 0)$ is a saddle. These are shown by open dots in the figure.

22. We have the model

$$x_1' = (a_1 - b_1x_1 + c_1x_2)x_1$$
$$x_2' = (a_2 - b_2x_2 + c_2x_1)x_2$$

where $a_1 < 0, b_1 = 0, c_1 > 0, a_2 > 0, b_2 > 0$, and $c_2 < 0$.

23. We have the model

$$x_1' = (a_1 - b_1x_1 + c_1x_2)x_1$$
$$x_2' = (a_2 - b_2x_2 + c_2x_1)x_2$$

where $a_1 < 0, b_1 = 0, c_1 > 0, a_2 < 0, b_2 = 0$, and $c_2 > 0$.

25. We have the model

$$x_1' = (a_1 - b_1x_1 + c_1x_2)x_1$$
$$x_2' = (a_2 - b_2x_2 + c_2x_1)x_2$$

where $a_1 > 0, b_1 > 0, c_1 > 0, a_2 > 0, b_2 > 0$, and $c_2 > 0$.

Section 10.2

1. The equilibrium points are $(0, 0)$, $(1, 1)$, and $(-1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 1 - y & -x \\ 2x & -1 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$(0, 0)$ is a saddle point. Since

$$J(1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$$

has determinant $D = 2$, trace $T = -1$, and discriminant $T^2 - 4D < 0$, $(1, 1)$ is a spiral sink. $J(-1, 1)$ has the same trace and determinant so it is also a spiral sink.

4. The equilibrium points are $(0, 0)$, $(1, -1)$, and $(-1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 1 & 1 \\ -2xy & 1 - x^2 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$(0, 0)$ is a nongeneric source and the equilibrium point cannot be classified. Since

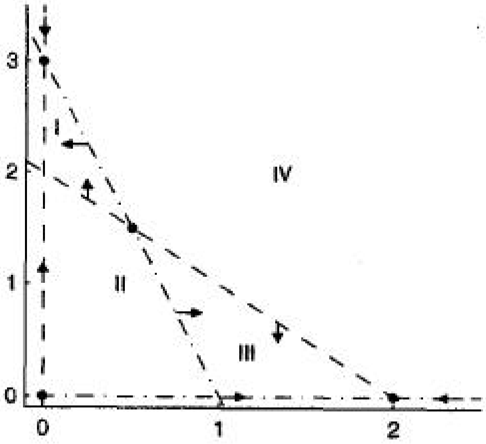
$$J(1, -1) = J(-1, 1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

has determinant $D = -2$, both $(1, -1)$ and $(-1, 1)$ are saddles.

Section 10.3

1. The x-axis is defined by $y = 0$. Notice that if $x(t)$ solves the logistic equation $x' = (2 - x)x$ and $y(t) = 0$ then x and y are a solution to the system in the exercise. Every point in the x-axis is contained in such a solution curve, which stays in the x-axis. Hence the x-axis is invariant. Similarly the functions $x(t) = 0$ and $y(t)$ defined by $y' = (3 - y)y$ are solutions to the system in the exercise, and their solution curves exhaust the y-axis, making it invariant. A solution curve starting in one of the four quadrants must stay in that quadrant, because to get out it has to cross one of the axes. It cannot do so because the unique solution curve through any point in the axis must be entirely contained in the axis.

9. The x-nullcline is the union of the two lines $x = 0$ and $x + y = 2$, shown dashed.



The y-nullcline is the union of the two lines $y = 0$ and $3x + y = 3$, shown dot-dashed. These intersect in the four equilibrium points $(0, 0)$, $(2, 0)$, $(0, 3)$, and $(1/2, 3/2)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -3y & 3 - 3x - 2y \end{pmatrix}.$$

At $(0, 0)$,

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

has eigenvalues 2 and 3, so the origin is a nodal source. At $(2, 0)$,

$$J(2, 0) = \begin{pmatrix} -2 & -2 \\ 0 & -3 \end{pmatrix},$$

has eigenvalues -2 and -3, so $(2, 0)$ is a nodal sink. At $(0, 3)$,

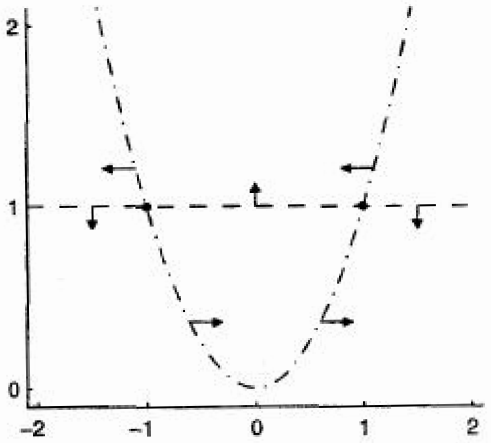
$$J(0, 3) = \begin{pmatrix} -1 & 0 \\ -9 & -3 \end{pmatrix},$$

has eigenvalues -1 and -3, so $(0, 3)$ is also a nodal sink. Finally,

$$J(1/2, 3/2) = \begin{pmatrix} -1/2 & -1/2 \\ -9/2 & -3/2 \end{pmatrix}$$

has determinant $D = -3/2 < 0$, so $(1/2, 3/2)$ is a saddle. The flow of the solutions along the nullclines is shown by the arrows. This information shows that regions I and III are invariant. The solution curves flow from region II into regions I and III, except the one stable solution curve for the saddle at $(1/2, 3/2)$. The solution curves in region IV can flow directly to the sinks, or into regions I and III, with the exception of the one stable solution curve for the saddle at $(1/2, 3/2)$. Finally solution curves in the invariant region I flow to the sink at $(0, 3)$ and the solution curves in the invariant region III flow to the sink at $(2, 0)$.

13. The x-nullcline is the line $y = 1$ shown dashed. The y-nullcline is the parabola $y = x^2$ shown dot-dashed.



These intersect in the equilibrium points $(\pm 1, 1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} 0 & -1 \\ -2x & 1 \end{pmatrix}.$$

Since

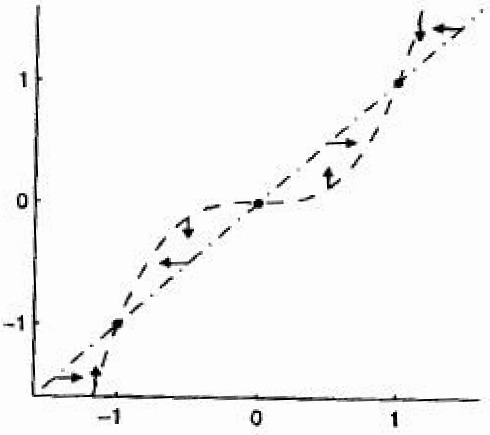
$$J(1, 1) = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$$

has determinant $D = -2$, so $(1, 1)$ is a saddle. Since

$$J(-1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$$

has determinant $D = 2$, trace $T = 1$, and discriminant $T^2 - 4D < 0$, $(-1, 1)$ is a spiral source. The flow of the solutions along the nullclines is shown by the arrows.

15. The x-nullcline is the cubic curve $y = x^3$ shown dashed. The y-nullcline is the line $x = y$ shown dot-dashed.



These intersect in three equilibrium points $(0, 0)$, $(1, 1)$ and $(-1, -1)$. The Jacobian is

$$J(x, y) = \begin{pmatrix} -3x^2 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

has determinant $D = -1$, the origin is a saddle. At the other two equilibrium points

$$J = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix},$$

which has determinant $D = 2$, trace $T = -4$, and discriminant $T^2 - 4D > 0$. Hence $(1, 1)$ and $(-1, -1)$ are nodal sinks. The flow of the solutions along the nullclines is shown by the arrows. The nullclines split all of $\mathbf{R}^2$ into six regions. Three of them are invariant. The flow of all solutions can be followed as they converge to one of the two sinks, except for the two stable solutions for the saddle point.

Section 5.1

2. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty -2e^{-st}\,dt\\ &= \lim_{T\to\infty}\int_0^T -2e^{-st}\,dt\\ &= \lim_{T\to\infty}\left.\frac{2e^{-st}}{s}\right|_0^T\\ &= \lim_{T\to\infty}\left[\frac{2e^{-st}}{s}-\frac{2}{s}\right]\\ &= -\frac{2}{s},\end{aligned}$$

provided, $s > 0$.

3. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty e^{-2t}e^{-st}\,dt\\ &= \lim_{T\to\infty}\int_0^T e^{-(s+2)t}\,dt\\ &= \lim_{T\to\infty}\left.\frac{-e^{-(s+2)t}}{s+2}\right|_0^T\\ &= \lim_{T\to\infty}\left[\frac{-e^{-(s+2)T}}{s+2}+\frac{1}{s+2}\right]\\ &= \frac{1}{s+2},\end{aligned}$$

provided $s > -2$.

25. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty f(t)e^{-st}\,dt\\ &= \int_0^2 0e^{-st}\,dt+\int_2^\infty 2e^{-st}\,dt\\ &= 2\lim_{T\to\infty}\left.\frac{-e^{-st}}{s}\right|_2^T\\ &= \frac{2e^{-2s}}{s},\end{aligned}$$

provided $s > 0$.

8. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty te^{-3t}e^{-st}\,dt\\ &= \lim_{T\to\infty}\int_0^T te^{-(s+3)t}\,dt\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int te^{-(s+3)t}\,dt = -&\frac{t}{s+3}e^{-(s+3)t}\\ &-\frac{1}{(s+3)^2}e^{-(s+3)t}.\end{aligned}$$

Thus,

$$\begin{aligned}F(s) &= \lim_{T\to\infty}\left[-\frac{t}{s+3}e^{-(s+3)t}\right.\\ &\qquad\left.-\frac{1}{(s+3)^2}e^{-(s+3)t}\right]_0^T\\ &= \frac{1}{(s+3)^2},\end{aligned}$$

provided $s > -3$.

28. Using Definition 1.1,

$$\begin{aligned}F(s) &= \int_0^\infty f(t)e^{-st}\,dt\\ &= \int_0^3 te^{-st}\,dt+\int_3^\infty 3e^{-st}\,dt\end{aligned}$$

Use integration by parts on the first integral.

$$\begin{aligned}\int_0^3 te^{-st}\,dt &= \left[-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\right]_0^3\\ &= -\frac{3}{s}e^{-3s}-\frac{1}{s^2}e^{-3s}+\frac{1}{s^2}\end{aligned}$$

The second integral is

$$\begin{aligned}\int_3^\infty 3e^{-st}\,dt &= \lim_{T\to\infty}\int_3^T 3e^{-st}\,dt\\ &= \lim_{T\to\infty}\left.-\frac{3}{s}e^{-st}\right|_3^T\\ &= \lim_{T\to\infty}\left[-\frac{3}{s}e^{-sT}+\frac{3}{s}e^{-3s}\right]\\ &= \frac{3}{s}e^{-3s}.\end{aligned}$$

Therefore,

$$\begin{aligned}F(s) &= -\frac{3}{s}e^{-3s}-\frac{1}{s^2}e^{-3s}+\frac{1}{s^2}+\frac{3}{s}e^{-3s}\\ &= -\frac{1}{s^2}e^{-3s}+\frac{1}{s^2},\end{aligned}$$

provided $s > 0$.