Section 5.5

1. Since $f(t) = t$ has transform $F(s) = 1/s^2$, $g(t) = H(t-2)(t-2)$ has transform

$$G(s) = e^{-2s}F(s) = e^{-2s}\cdot\frac{1}{s^2} = \frac{e^{-2s}}{s^2}.$$

5. Since $(t-1)^2 = t^2 - 2t + 1$, we can write

$$t^2 = (t-1)^2 + 2t - 1 = (t-1)^2 + 2(t-1) + 1.$$

Thus,

$$H(t-1)t^2 = H(t-1)(t-1)^2 + 2H(t-1)(t-1) + H(t-1),$$

and

$$\begin{aligned}\mathcal{L}\{H(t-1)t^2\}(s) &= \mathcal{L}\{H(t-1)(t-1)^2\}(s) + 2\mathcal{L}\{H(t-1)(t-1)\}(s) + \mathcal{L}\{H(t-1)\}(s)\\ &= e^{-s}\cdot\frac{2!}{s^3} + 2e^{-s}\cdot\frac{1}{s^2} + e^{-s}\cdot\frac{1}{s}\\ &= \frac{e^{-s}(2+2s+s^2)}{s^3}.\end{aligned}$$

13. The function

$$f(t) = \begin{cases} 0, & \text{if } t < 0,\\ t^2, & \text{if } 0 \le t < 2,\\ 4, & \text{if } t \ge 2,\end{cases}$$

can be written

$$\begin{aligned}f(t) &= t^2H_{02}(t) + 4H_2(t)\\ &= t^2(H(t) - H(t-2)) + 4H(t-2)\\ &= t^2H(t) - t^2H(t-2) + 4H(t-2)\\ &= t^2H(t) - (t-2+2)^2H(t-2) - 4H(t-2)\\ &= t^2H(t) - (t-2)^2H(t-2) - 4(t-2)H(t-2).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{t^2H(t)\}(s) - \mathcal{L}\{(t-2)^2H(t-2)\}(s) - 4\mathcal{L}\{(t-2)H(t-2)\}(s)\\ &= \frac{2!}{s^3} - e^{-2s}\cdot\frac{2!}{s^3} - 4e^{-2s}\cdot\frac{1}{s^2}\\ &= \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2}.\end{aligned}$$

22. A partial fraction decomposition.

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$
$$1 = A(s+2) + Bs$$

Then,

$$s = -2 \Rightarrow B = -\frac{1}{2}$$
$$s = 0 \Rightarrow A = \frac{1}{2}$$

and

$$\frac{1-e^{-s}}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)} - \frac{e^{-s}}{2s} + \frac{e^{-s}}{2(s+2)}.$$

Note the transform pairs.

$$f(t) = 1 \leftrightarrow F(s) = \frac{1}{s}$$
$$g(t) = e^{-2t} \leftrightarrow G(s) = \frac{1}{s+2}$$

Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{2s} - \frac{1}{2(s+2)} - e^{-s}\left[\frac{1}{2s} - \frac{1}{2(s+2)}\right]\right\}(t)\\ &= \frac{1}{2}\left[\mathcal{L}^{-1}\{F(s)\}(t) - \mathcal{L}^{-1}\{G(s)\}(t) - \mathcal{L}^{-1}\{e^{-s}F(s)\}(t) + \mathcal{L}^{-1}\{e^{-s}G(s)\}(t)\right]\\ &= \frac{1}{2}\left[1 - e^{-2t} - H(t-1)f(t-1) + H(t-1)g(t-1)\right]\\ &= \frac{1}{2}\left[1 - e^{-2t} - H(t-1) + H(t-1)e^{-2(t-1)}\right]\end{aligned}$$

28. The forcing function f is described by

$$f(t) = \begin{cases} 1, & \text{if } 1 \le t < 2, \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently,

$$\begin{aligned} f(t) &= H_{12}(t) \\ &= H(t-1) - H(t-2). \end{aligned}$$

Hence,

$$F(s) = \mathcal{L}\{f(t)\}(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Take the Laplace transform of both sides of the given equation. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$\begin{aligned} y'' + 4y &= f(t) \\ \mathcal{L}(y'')(s) + 4\,\mathcal{L}(y)(s) &= \mathcal{L}\{f(t)\}(s) \\ s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= F(s) \end{aligned}$$

Use the initial conditions $y(0) = y'(0) = 0$ and the Laplace transform of f found earlier.

$$\begin{aligned} s^2Y(s) + 4Y(s) &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\ Y(s) &= \frac{1}{s^2+4}\left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right] \\ Y(s) &= \frac{e^{-s}}{s(s^2+4)} - \frac{e^{-2s}}{s(s^2+4)} \end{aligned}$$

A partial fraction decomposition is needed.

$$\begin{aligned} \frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} \\ 1 &= A(s^2+4) + (Bs+C)s \\ 1 &= (A+B)s^2 + Cs + 4A \end{aligned}$$

Then,

$$\begin{aligned} s = 0 &\Rightarrow A = \frac{1}{4} \\ A + B = 0 &\Rightarrow B = -\frac{1}{4} \\ C &= 0. \end{aligned}$$

Therefore,

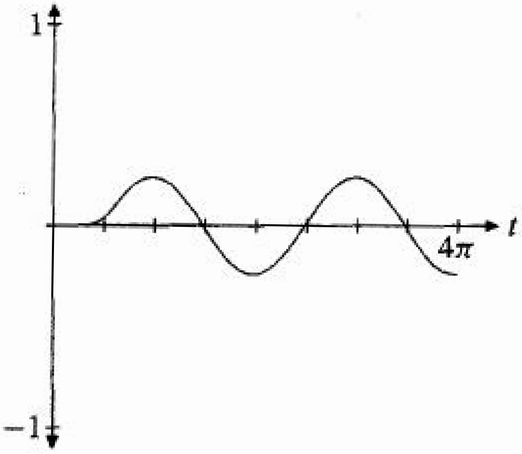
$$\frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{s}{4(s^2+4)},$$

which has transform

$$\frac{1}{4} - \frac{1}{4}\cos 2t.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}(t) \\ &\quad - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+4)}\right\}(t) \\ &= H(t-1)\left[\frac{1}{4} - \frac{1}{4}\cos(t-1)\right] \\ &\quad - H(t-2)\left[\frac{1}{4} - \frac{1}{4}\cos(t-2)\right]. \end{aligned}$$



Section 5.6

2. By Theorem 6.10, the unit impulse response of

$$y'' - 4y' + 3y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 - 4s + 3$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 - 4s + 3}.$$

Using a partial fraction decomposition,

$$E(s) = \frac{1}{2} \cdot \frac{1}{s-3} - \frac{1}{2} \cdot \frac{1}{s-1},$$

so the unit impulse response is

$$e(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{t}.$$

5. By Theorem 6.10, the unit impulse response of

$$y'' - 9y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 - 9$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 - 9}.$$

Using a partial fraction decomposition,

$$E(s) = -\frac{1}{6} \cdot \frac{1}{s+3} + \frac{1}{6} \cdot \frac{1}{s-3},$$

so the unit impulse response is

$$e(t) = -\frac{1}{6}e^{-3t} + \frac{1}{6}e^{3t}.$$

Section 5.7

8. If $f(t) = t$ and $g(t) = e^t$, then

$$\begin{aligned} f * g(t) &= \int_0^t f(u)g(t-u)\,du, \\ &= \int_0^t u(e^{t-u})\,du. \end{aligned}$$

Integration by parts provides

$$f * g(t) = -ue^{t-u}\Big|_0^t + \int_0^t e^{t-u}\,du.$$

A second integration yields

$$\begin{aligned} &= -ue^{t-u}\Big|_0^t - e^{t-u}\Big|_0^t, \\ &= -te^0 - e^0 + e^t, \\ &= -t - 1 + e^t. \end{aligned}$$

22. The expression

$$\begin{aligned} \frac{1}{(s+1)(s^2+4)} &= \frac{1}{s+1} \cdot \frac{1}{s^2+4}, \\ &= F(s)G(s). \end{aligned}$$

We have transform pairs

$$\begin{aligned} F(s) = \frac{1}{s+1} &\Longleftrightarrow f(t) = e^{-t}, \\ G(s) = \frac{1}{2} \cdot \frac{2}{s^2+4} &\Longleftrightarrow g(t) = \frac{1}{2}\sin 2t. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\} \\ &= \mathcal{L}^{-1}\{F(s)G(s)\}(t), \\ &= f * g(t), \\ &= \int_0^t f(u)g(t-u)\,du, \\ &= \frac{1}{2}\int_0^t e^{-u}\sin 2(t-u)\,du. \end{aligned}$$

Two integrations by parts show that

$$\begin{aligned} &\int e^{-u}\sin(2t-2u)\,du \\ &= \frac{1}{5}e^{-u}\left[2\cos(2t-2u) - \sin(2t-2u)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\}(t) \\ &= \frac{1}{10}e^{-u}\left[2\cos(2t-2u) - \sin(2t-2u)\right]\Big|_0^t \\ &= \frac{1}{5}e^{-t} + \frac{1}{10}\sin 2t - \frac{1}{5}\cos 2t. \end{aligned}$$

27. We start by computing the impulse response functic $e(t)$.

$$e'' + 9e = \delta(t), \qquad e(0) = e'(0) = 0.$$

By Theorem 6.10, the Laplace transform of $e(t)$ is

$$E(s) = \frac{1}{P(s)} = \frac{1}{s^2+9}.$$

Thus,

$$\begin{aligned} e(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}(t) \\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}(t) \\ &= \frac{1}{3}\sin 3t. \end{aligned}$$

Compute the derivative.

$$e'(t) = \cos 3t$$

From Theorem 7.16, we know that the solution of

$$y'' + 9y = g(t), \qquad y(0) = -1, y'(0) = 2$$

is

$$\begin{aligned} y(t) &= e * g(t) + ay_0e'(t) \\ &\quad + (ay_1 + by_0)e(t) \\ &= e * g(t) + (1)(-1)e'(t) \\ &\quad + ((1)(2) + (0)(-1))e(t) \\ &= e * g(t) - e'(t) + 2e(t) \\ &= \frac{1}{3}\int_0^t (\sin 3u)g(t-u)\,du \\ &\quad - \cos 3t + \frac{2}{3}\sin 3t. \end{aligned}$$

29. We start by computing the impulse response function $e(t)$.

$$e'' + 4e' + 3e = \delta(t), \qquad e(0) = e'(0) = 0.$$

By Theorem 6.10, the Laplace transform of $e(t)$ is

$$\begin{aligned} E(s) &= \frac{1}{P(s)} \\ &= \frac{1}{s^2+4s+3} \\ &= \frac{1}{(s+1)(s+3)} \\ &= \frac{1/2}{s+1} - \frac{1/2}{s+3}. \end{aligned}$$

Thus,

$$e(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

Compute the derivative.

$$e'(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}$$

From Theorem 7.16, we know that the solution of

$$y'' + 4y' + 3y = g(t), \quad y(0) = -1, y'(0) = 1$$

is

$$\begin{aligned} y(t) &= e * g(t) + ay_0e'(t) + (ay_1 + by_0)e(t) \\ &= e * g(t) + (1)(-1)e'(t) \\ &\quad + ((1)(1) + (4)(-1))e(t) \\ &= e * g(t) - e'(t) - 3e(t) \\ &= \int_0^t \left(\frac{1}{2}e^{-u} - \frac{1}{2}e^{-3u}\right) g(t-u)\,du \\ &\quad + \frac{1}{2}e^{-t} - \frac{3}{2}e^{-3t} - \frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \\ &= \int_0^t \left(\frac{1}{2}e^{-u} - \frac{1}{2}e^{-3u}\right) g(t-u)\,du \\ &\quad - e^{-t}. \end{aligned}$$