Section 5.2

4. Using Proposition 2.7 and Table 1,

$$\begin{aligned}&\mathcal{L}(3-5t-11t^3)(s)\\&=3\,\mathcal{L}(1)(s)-5\,\mathcal{L}(t)(s)-11\,\mathcal{L}(t^3)(s)\\&=3\cdot\frac{1}{s}-5\cdot\frac{1!}{s^2}-11\cdot\frac{3!}{s^4}\\&=\frac{3}{s}-\frac{5}{s^2}-\frac{66}{s^4},\end{aligned}$$

provided $s>0$.

19. If $y'-5y=e^{-2t}$, with $y(0)=1$, then

$$\begin{aligned}\mathcal{L}\{y'-5y\}(s)&=\mathcal{L}\{e^{-2t}\}(s)\\ \mathcal{L}\{y'\}(s)-5\,\mathcal{L}\{y\}(s)&=\frac{1}{s+2}\\ s\,\mathcal{L}\{y\}(s)-y(0)-5\,\mathcal{L}\{y\}(s)&=\frac{1}{s+2}.\end{aligned}$$

If we let $Y(s)=\mathcal{L}\{y\}(s)$, then

$$\begin{aligned}sY(s)-1-5Y(s)&=\frac{1}{s+2}\\(s-5)Y(s)&=1+\frac{1}{s+2}\\Y(s)&=\frac{1}{s-5}+\frac{1}{(s-5)(s+2)}\\Y(s)&=\frac{(s+2)+1}{(s-5)(s+2)}\\Y(s)&=\frac{s+3}{(s-5)(s+2)}.\end{aligned}$$

24. If $y''+y'+2y=\cos 2t+\sin 3t$, with $y(0)=-1$, and $y'(0)=1$, then,

$$\begin{aligned}&[s^2\,\mathcal{L}(y)(s)-sy(0)-y'(0)]\\&\qquad+[s\,\mathcal{L}(y)(s)-y(0)]+2\,\mathcal{L}(y)(s)\\&=\mathcal{L}(\cos 2t)+\mathcal{L}(\sin 3t)\\&=\frac{s}{s^2+2^2}+\frac{3}{s^2+3^2}.\end{aligned}$$

Letting $Y(s)=\mathcal{L}(y)(s)$, and solving for $Y(s)$,

$$\begin{aligned}&(s^2+s+2)Y(s)\\&=-s+\frac{s}{s^2+4}+\frac{3}{s^2+9}\\&=\frac{-s^5-13s^3-36s+s^3+9s+3s^2+12}{(s^2+4)(s^2+9)}\end{aligned}$$

or

$$Y(s)=\frac{-s^5-12s^3+3s^2-27s+12}{(s^2+s+2)(s^2+4)(s^2+9)}.$$

27. Because $f(t)=\cos 2t$ has transform $F(s)=s/(s^2+4)$, then $y(t)=e^{2t}\cos 2t$ will have transform

$$\begin{aligned}Y(s)&=F(s-2)\\&=\frac{s-2}{(s-2)^2+4}\\&=\frac{s-2}{s^2-4s+8}.\end{aligned}$$

32. Consider the transform pair,

$$y(t)=\cos 2t\leftrightarrow Y(s)=\frac{s}{s^2+4}.$$

By Proposition 2.14,

$$\begin{aligned}&\mathcal{L}\{t^2\cos 2t\}(s)\\&=(-1)^2Y''(s)\\&=\frac{d}{ds}\left[\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right]\\&=\frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]\\&=\frac{(s^2+4)^2(-2s)-(4-s^2)(2)(s^2+4)(2s)}{(s^2+4)^4}\\&=\frac{-2s(s^2+4)[(s^2+4)+2(4-s^2)]}{(s^2+4)^4}\\&=\frac{-2s(12-s^2)}{(s^2+4)^3}.\end{aligned}$$

Section 5.3

1. Factor.

$$Y(s) = \frac{1}{3s+2} = \frac{1}{3} \cdot \frac{1}{s+2/3}.$$

Now,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{3} \cdot \frac{1}{s+2/3}\right\} \\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2/3}\right\} \\ &= \frac{1}{3}e^{-(2/3)t}. \end{aligned}$$

8. Adjust as follows.

$$\begin{aligned} Y(s) &= \frac{2-5s}{s^2+9} \\ &= \frac{2}{s^2+9} - \frac{5s}{s^2+9} \\ &= \frac{2}{3} \cdot \frac{3}{s^2+9} - 5 \cdot \frac{s}{s^2+9} \end{aligned}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} - 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= \frac{2}{3}\sin 3t - 5\cos 3t. \end{aligned}$$

14. Note the transform pair.

$$\cos 2t \leftrightarrow \frac{s}{s^2+4}$$

By Proposition 2.12,

$$e^t \cos 2t \leftrightarrow \frac{s-1}{(s-1)^2+4}.$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{4(s-1)}{(s-1)^2+4}\right\} \\ &= 4\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} \\ &= 4e^t \cos 2t. \end{aligned}$$

19. Find a partial fraction decomposition.

$$\begin{aligned} \frac{1}{(s+2)(s-1)} &= \frac{A}{s+2} + \frac{B}{s-1} \\ 1 &= A(s-1) + B(s+2) \end{aligned}$$

Now,

$$s = 1 \Rightarrow 1 = 3B \text{ or } B = \frac{1}{3}$$

$$s = -2 \Rightarrow 1 = -3A \text{ or } A = -\frac{1}{3}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right\} \\ &= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t. \end{aligned}$$

24. Find a partial fraction decomposition.

$$\begin{aligned} \frac{7-s}{s^2+s-2} &= \frac{A}{s+2} + \frac{B}{s-1} \\ 7-s &= A(s-1) + B(s+2) \end{aligned}$$

Now,

$$s = 1 \Rightarrow 6 = 3B \text{ or } B = 2$$

$$s = -2 \Rightarrow 9 = -3A \text{ or } A = -3.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{-3}{s+2} + \frac{2}{s-1}\right\} \\ &= -3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= -3e^{-2t} + 2e^t. \end{aligned}$$

Section 5.4

3. Set $Y = \mathcal{L}(y)$. Then,

$$\begin{aligned}\mathcal{L}(y' + 4y) &= \mathcal{L}(\cos t)\\ \mathcal{L}(y') + 4\,\mathcal{L}(y) &= \frac{s}{s^2+1}\\ s\cdot Y(s) - y(0) + 4Y(s) &= \frac{s}{s^2+1}.\end{aligned}$$

But $y(0) = 0$, so

$$\begin{aligned}(s+4)Y(s) &= \frac{s}{s^2+1}\\ Y(s) &= \frac{s}{(s+4)(s^2+1)}.\end{aligned}$$

Find a partial fraction decomposition.

$$\frac{s}{(s+4)(s^2+1)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+1}$$

Equating numerators

$$\begin{aligned}s &= A(s^2+1) + (Bs+C)(s+4)\\ &= (A+B)s^2 + (4B+C)s + (A+4C).\end{aligned}$$

Thus,

$$\begin{aligned}A + B &= 0\\ 4B + C &= 1\\ A + 4C &= 0,\end{aligned}$$

and $A = -4/17$, $B = 4/17$, and $C = 1/17$. Thus,

$$\begin{aligned}Y(s) &= \frac{-4/17}{s+4} + \frac{(4/17)s + 1/17}{s^2+1}\\ &= -\frac{4}{17}\cdot\frac{1}{s+4} + \frac{4}{17}\cdot\frac{s}{s^2+1}\\ &\qquad + \frac{1}{17}\cdot\frac{1}{s^2+1},\end{aligned}$$

and

$$y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos t + \frac{1}{17}\sin t.$$

5. Set $Y = \mathcal{L}(y)$. Then,

$$\begin{aligned}\mathcal{L}(y' + 6y) &= \mathcal{L}(2t + 3)\\ \mathcal{L}(y') + 6\,\mathcal{L}(y) &= 2\,\mathcal{L}(t) + 3\,\mathcal{L}(1)\\ s\cdot Y(s) - y(0) + 6Y(s) &= \frac{2}{s^2} + \frac{3}{s}.\end{aligned}$$

But $y(0) = 1$, so

$$(s+6)Y(s) - 1 = \frac{2}{s^2} + \frac{3}{s}.$$

Solving for Y,

$$\begin{aligned}Y(s) &= \frac{1}{s+6} + \frac{2}{s^2(s+6)} + \frac{3}{s(s+6)}\\ &= \frac{1}{s+6} + \frac{3s+2}{s^2(s+6)}.\end{aligned}$$

Find a partial fraction decomposition.

$$\frac{3s+2}{s^2(s+6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s+6}$$

Equating numerators,

$$\begin{aligned}3s + 2 &= As(s+6) + B(s+6) + (Cs+D)s^2\\ 3s + 2 &= Cs^3 + (A+D)s^2 + (6A+B)s + 6B.\end{aligned}$$

Thus,

$$\begin{aligned}C &= 0\\ A + D &= 0\\ 6A + B &= 3\\ 6B &= 2,\end{aligned}$$

and $A = 4/9$, $B = 1/3$, $C = 0$, and $D = -4/9$. Thus,

$$\begin{aligned}Y(s) &= \frac{1}{s+6} + \frac{4/9}{s} + \frac{1/3}{s^2} - \frac{4/9}{s+6}\\ &= \frac{5/9}{s+6} + \frac{4/9}{s} + \frac{1/3}{s^2}.\end{aligned}$$

Therefore,

$$y(t) = \frac{5}{9}e^{-6t} + \frac{4}{9} + \frac{1}{3}t.$$

19. Let $\mathcal{L}(y) = Y$. Since $y(0) = -1$ and $y'(0) = 0$,

$$\mathcal{L}(y''-4y'-5y) = s^2Y(s)+s-4[sY(s)+1]-5Y(s)$$

In addition,

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}.$$

Solving

$$(s^2-4s-5)Y(s)+s-4 = \frac{1}{s-2}$$

for Y, we get

$$Y(s) = \frac{-s+4}{s^2-4s-5} + \frac{1}{(s-2)(s^2-4s-5)}.$$

The terms on the right have decompositions:

$$\frac{-s+4}{s^2-4s-5} = \frac{-5/6}{s+1} - \frac{1/6}{s-5}$$

$$\frac{1}{(s-2)(s^2-4s-5)} = \frac{-1/9}{s-2} + \frac{1/18}{s+1} + \frac{1/18}{s-5}.$$

Thus,

$$Y(s) = \frac{-7/9}{s+1} - \frac{1/9}{s-5} - \frac{1/9}{s-2},$$

and

$$y(t) = -\frac{7}{9}e^{-t} - \frac{1}{9}e^{5t} - \frac{1}{9}e^{2t}.$$

25. Let $\mathcal{L}(y) = Y$. Since $y(0) = 1$ and $y'(0) = 0$,

$$\mathcal{L}(y''+4y) = s^2Y(s) - s + 4Y(s).$$

On the other hand, we have the transform pair

$$\cos t \Longleftrightarrow \frac{s}{s^2+1},$$

so by Proposition 2.12,

$$e^{-2t}\cos 2t \Longleftrightarrow \frac{s+2}{(s+2)^2+1}.$$

Thus,

$$(s^2+4)Y(s) - s = \frac{s+2}{(s+2)^2+1}, \qquad (4.2)$$

so

$$Y(s) = \frac{s}{s^2+4} + \frac{s+2}{(s^2+4)((s+2)^2+1)}. \qquad (4.3)$$

The fraction on the right has decomposition

$$\frac{s+2}{(s^2+4)((s+2)^2+1)} = -\frac{1}{65}\cdot\frac{7s-18}{s^2+4} + \frac{1}{65}\cdot\frac{7s+10}{(s+2)^2+1}.$$

Thus,

$$\begin{aligned} Y(s) &= \frac{1}{65}\cdot\frac{58s+18}{s^2+4} + \frac{1}{65}\cdot\frac{7s+10}{(s+2)^2+1} \\ &= \frac{1}{65}\left[58\cdot\frac{s}{s^2+4} + 9\cdot\frac{2}{s^2+4}\right] \\ &\quad + \frac{1}{65}\left[7\cdot\frac{s+2}{(s+2)^2+1} - 4\cdot\frac{1}{(s+2)^2+1}\right] \end{aligned}$$

The transform pairs

$$\cos t \Longleftrightarrow \frac{s}{s^2+1}$$

$$\sin t \Longleftrightarrow \frac{1}{s^2+1}$$

and Proposition 2.12 yield

$$e^{-2t}\cos t \Longleftrightarrow \frac{s+2}{(s+2)^2+1}$$

$$e^{-2t}\sin t \Longleftrightarrow \frac{1}{(s+2)^2+1}.$$

Therefore,

$$\begin{aligned} y(t) =& \frac{58}{65}\cos 2t + \frac{9}{65}\sin 2t \\ &+ \frac{7}{65}e^{-2t}\cos t - \frac{4}{65}e^{-2t}\sin t. \end{aligned}$$