Section 2.3

3. The depth of the well satisfies $d = 4.9t^2$, where t is the amount of time it takes the stone to hit the water. It also satisfies $d = 340s$, where $s = 8 - t$ is the amount of time it takes for the noise of the splash to reach the ear. Thus we must solve the quadratic equation $4.9t^2 = 340(8-t)$. The solution is $t = 7.2438$sec. The depth is $d = 340(8-t) = 257.1$m.

4. In the first 60s the rocket rises to an elevation of $(100-9.8)t^2/2 = 162,360$m and achieves a velocity of $v(60) = (100-9.8)*60 = 5412$m/s. After that the velocity is $5412 - 9.8t$. This is zero at the highest point, reached when $t_1 = 552.2$s. The altitude at that point is $162,360 + 5412t_1 - 9.8t_1^2/2 = 1.657 \times 10^6$m. From there to the ground it takes t_2s, where $4.9t_2^2 = 1.657 \times 10^6$, or $t_2 = 581.5$s. The total trip takes $60 + 552.2 + 581.5 = 1193.7$s.

8. We have $v(t) = Ce^{-rt/m} - mg/r$. If $v(0) = 0$ then $C = mg/r$, and $v(t) = mg(e^{-rt/m} - 1)/r$. This is equal to $-mg/2r$ when $e^{-rt/m} = 1/2$. Thus the time required is $t = m\ln(2)/r$. The distance traveled is

$$\begin{aligned} x &= \int_0^t v(s)\,ds \\ &= \frac{mg}{r}\int_0^t (e^{-rs/m} - 1)\,ds \\ &= \frac{mg}{r}\left[\frac{m}{r}(1 - e^{-rt/m}) - t\right] \\ &= \frac{mg}{r}\left[\frac{m}{2r} - \frac{m\ln 2}{r}\right] \\ &= \frac{m^2 g}{r^2}\left[\frac{1}{2} - \ln 2\right] \end{aligned}$$

9. The resistance force has the form $R = -rv$. When $v = 0.2$, $R = -1$ so $r = 5$. The terminal velocity is $v_{\text{term}} = -mg/r = -0.196$m/s.

10. (a) First, the terminal velocity gives us $20 = mg/r$, or $r = mg/20 = 70 \times 9.8/20 = 34.3$. Next, we have $v(t) = Ce^{-rt/m} - mg/r$. Since $v(0) = 0$, $C = mg/r$, and $v(t) = mg(e^{-rt/m} - 1)/r$. Integrating and setting $x(0) = 0$, we get

$$\begin{aligned} x &= \int_0^t v(s)\,ds \\ &= \frac{mg}{r}\int_0^t (e^{-rs/m} - 1)\,ds \\ &= \frac{mg}{r}\left[\frac{m}{r}(1 - e^{-rt/m}) - t\right] \end{aligned}$$

Hence $v(2) = -12.4938$ and $x(2) = -14.5025$.

(b) The velocity is 80% of its terminal velocity when $1 - e^{-rt/m} = 0.8$. For the values of $m = 70$ and $r = 34.3$ this becomes $t = 3.2846$s.

Section 2.4

2. We have $a(t) = 3$, so $u(t) = e^{-3t}$. Multiplying we see that the equation becomes

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}.$$

We verify that the left-hand side is the derivative of $e^{-3t}y$, so when we integrate we get

$$e^{-3t}y(t) = -\frac{5}{3}e^{-3t} + C.$$

Solving for y, we get

$$y(t) = -\frac{5}{3} + Ce^{3t}.$$

6. If we write the equations as $x' = (4/t)x + t^3$, we see that $a(t) = 4/t$. Thus the integrating factor is

$$u(t) = e^{-\int (4/t)\,dt} = e^{-4\ln t} = t^{-4}.$$

Multiplying by u, the equation becomes

$$t^{-4}x' - 4t^{-5}x = t^{-1}.$$

After verifying that the left-hand side is the derivative of $t^{-4}x$, we can integrate and get

$$t^{-4}x(t) = \ln t + C.$$

Hence the general solution is

$$x(t) = t^4\ln t + Ct^4.$$

13. (a) Compare $y' + y\cos x = \cos x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\cos x$. Consequently, an integrating factor is found with

$$u(x) = e^{-\int a(x)\,dx} = e^{\int \cos x\,dx} = e^{\sin x}.$$

Multiply both sides of the differential equation by the integrating factor and check that the resulting left-hand side is the derivative of a product.

$$\begin{aligned} e^{\sin x}(y' + y\cos x) &= e^{\sin x}\cos x \\ \left(e^{\sin x}y\right)' &= e^{\sin x}\cos x \end{aligned}$$

Integrate and solve for y.

$$\begin{aligned} e^{\sin x}y &= e^{\sin x} + C \\ y(x) &= 1 + Ce^{-\sin x} \end{aligned}$$

(b) Separate the variables and integrate.

$$\begin{aligned} \frac{dy}{dx} &= \cos x(1-y) \\ \frac{dy}{1-y} &= \cos x\,dx \\ -\ln|1-y| &= \sin x + C. \end{aligned}$$

Take the exponential of each side.

$$\begin{aligned} |1-y| &= e^{-\sin x - C} \\ 1 - y &= \pm e^{-C}e^{-\sin x} \end{aligned}$$

If we let $A = \pm e^{-C}$, then

$$y(x) = 1 - Ae^{-\sin x},$$

where A is any real number, except zero. However, when we separated the variables above by dividing by $y - 1$, this was a valid operation only if $y \neq 1$. This hints at another solution. Note that $y = 1$ easily checks in the original equation. Consequently,

$$y(x) = 1 - Ae^{-\sin x},$$

where A is any real number. Note that this will produce the same solutions a $y = 1 + Ce^{-\sin x}$, C any real number, the solution found in part (a).

15. Solve for y'.

$$y' = -\frac{3x}{x^2+1}y + \frac{6x}{x^2+1}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -3x/(x^2+1)$. Consequently, an integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x)\,dx} = e^{\int 3x/(x^2+1)\,dx} \\ &= e^{(3/2)\ln(x^2+1)} = (x^2+1)^{3/2}. \end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (x^2+1)^{3/2}y' + 3x(x^2+1)^{1/2}y &= 6x(x^2+1)^{1/2} \\ \left((x^2+1)^{3/2}y\right)' &= 6x(x^2+1)^{1/2} \end{aligned}$$

Integrate and solve for y.

$$\begin{aligned} (x^2+1)^{3/2}y &= 2(x^2+1)^{3/2} + C \\ y &= 2 + C(x^2+1)^{-3/2} \end{aligned}$$

The initial condition gives

$$-1 = y(0) = 2 + C(0^2+1)^{-3/2} = 2 + C.$$

Therefore, $C = -3$ and $y(x) = 2 - 3(x^2+1)^{-3/2}$.

19. Solve for y'.

$$y' = \frac{1}{2x+3}y + (2x+3)^{-1/2}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = 1/(2x+3)$ and $f(x) = (2x+3)^{-1/2}$. It is important to note that a is continuous everywhere except $x = -3/2$, but f is continuous only on $(-3/2, +\infty)$, facts that will heavily influence our interval of existence.

An integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x)\,dx} = e^{\int -1/(2x+3)\,dx} \\ &= e^{-(1/2)\ln|2x+3|} = |2x+3|^{-1/2}. \end{aligned}$$

However, we will assume that $x > -3/2$ (a domain where both a and f are defined), so $u(x) = (2x+3)^{-1/2}$. Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (2x+3)^{-1/2}y' - (2x+3)^{-3/2}y &= (2x+3)^{-1} \\ \left((2x+3)^{-1/2}y\right)' &= (2x+3)^{-1} \end{aligned}$$

Integrate and solve for y.

$$(2x+3)^{-1/2}y = \frac{1}{2}\ln(2x+3) + C,$$

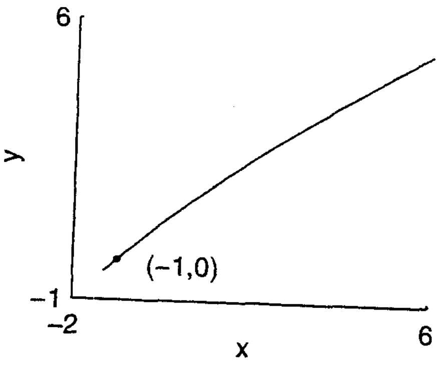
or

$$y = \frac{1}{2}(2x+3)^{1/2}\ln(2x+3) + C(2x+3)^{1/2}$$

The initial condition provides

$$0 = y(-1) = C.$$

Consequently, $y = (1/2)(2x+3)^{1/2}\ln(2x+3)$. The interval of existence is $(-3/2, +\infty)$ and the solution curve is shown in the following figure.



Section 2.5

1. (a) Let $S(t)$ denote the amount of sugar in the tank, measured in pounds. The rate in is 3 gal/min $\times$ 0.2 lb/gal $=$ 0.6 lb/min. The rate out is 3 gal/min $\times$ $S/100$ lb/gal $=$ $3S/100$ lb/min. Hence

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}$$
$$= 0.6 - 3S/100$$

This linear equation can be solved using the integrating factor $u(t) = e^{3t/100}$ to get the general solution $S(t) = 20 + Ce^{-3t/100}$. Since $S(0) = 0$, the constant $C = -20$ and the solution is $S(t) = 20(1 - e^{-3t/100})$.
$S(20) = 10(1 - e^{-0.6}) \approx 9.038$lb.

(b) $S(t) = 15$ when $e^{-3t/100} = 1 - 15/20 = 1/4$. Taking logarithms this translates to $t = (100\ln 4)/3 \approx 46.2098$m.

(c) As $t \to \infty$ $S(t) \to 20$.

6. The volume in the tank is decreasing at 1 gal/min, so the volume is $V(t) = 100 - t$. There is no sugar coming in, and the rate out is 3 gal/min $\times$ $S(t)/V(t)$ lb/gal. Hence the differential equation is

$$\frac{dS}{dt} = \frac{-3S}{100-t}.$$

This equation is linear and homogeneous. It can be solved by separating variables. The general solution is $S(t)+A(100-t)^3$. Since $S(0) = 100 \times 0.05 = 5$, we see that $A = 5 \times 10^{-6}$, and the solution is $S(t) = 5 \times 10^{-6} \times (100-t)^3$.
When $V(t) = 100 - t = 50$ gal,

$$S(t) = 5 \times 10^{-6} \times 50^3 = 0.625\text{ lb}.$$

7. (a) The volume of liquid in the tank is increasing by 2 gal/min. Hence the volume is $V(t) = 100 + 2t$ gal. Let $x(t)$ be the amount of pollutant in the tank, measured in lbs. The rate in during this initial period is 6 gal/min $\cdot$ 0.5 lb/gal $=$ 3 lb/gal. The rate out is 8 gl/min $\cdot x/V = 4x/(50+t)$. Hence the model equation is

$$x' = 3 - 4x/(50+t).$$

This linear equation can be solved using the integrating factor $u(t) = (50+t)^4$. The general solution is $x(t) = 3(50+t)/5 + C(50+t)^{-4}$. The initial condition $x(0) = 0$ allows us to compute the constant to be $C = -1.875 \times 10^8$. Hence the solution is

$$x(t) = \frac{3t}{5} + 30 - \frac{1.875 \times 10^8}{(50+t)^4}.$$

After 10 minutes the tank contains $x(10) = 21.5324$ lb of salt.

(b) Now the volume is decreasing at the rate of 4 gal/min from the initial volume of 120 gal. Hence if we start with $t = 0$ at the 10 minute mark, the volume is $V(t) = 120 - 4t$ gal. Now the rate in is 0, and the rate out is 8 gal/min $\cdot x/V = 2x/(30-t)$. Hence the model equation is

$$x' = -\frac{2x}{30-t}.$$

This homogeneous linear equation can solved by separating variables to find the general solution $x(t) = A(30-t)^2$. At $t = 0$ we have $x(0) = 21.5342$, from which we find that $A = 21.5342/900$, and the solution is

$$x(t) = \frac{21.5342}{900}(30-t)^2.$$

We are asked to find when this is one-half of 21.5342. This happens when $(30-t)^2 = 450$ or at $t = 8.7868$ min.