Section 1.1

1. Let $y(t)$ be the number of bacteria at time t. The rate of change of the number of bacteria is $y'(t)$. Since this rate of change is given to be proportional to $y(t)$, the resulting differential equation is $y'(t) = ky(t)$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of bacteria is growing).

2. Let $y(t)$ be the number of field mice at time t. The rate of change of the number of mice is $y'(t)$. Since this rate of change is given to be inversely proportional to the square root of $y(t)$, the resulting differential equation is $y'(t) = k/\sqrt{y(t)}$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of mice is growing).

3. Let $y(t)$ be the number of ferrets at time t. The rate of change of the number of ferrets is $y'(t)$. Since this rate of change is given to be proportional to the product of $y(t)$ and the difference between the maximum population and $y(t)$ (i.e. $100 - y(t)$), the resulting differential equation is $y'(t) = ky(t)(100 - y(t))$. Note that k is a positive constant since $y'(t)$ must be positive (i.e. the number of ferrets is growing provided $y(t) < 100$).

4. Let $y(t)$ be the quantity of radioactive substance at time t. The rate of change of the material is $y'(t)$. Since this rate of change (decay) is given to be proportional to $y(t)$, the resulting differential equation is $y'(t) = -ky(t)$. Note that k is a positive constant since $y'(t)$ must be negative (i.e. the quantity of radioactive material is decreasing).

6. Let $y(t)$ be the temperature of the potato at time t. The rate of change of the temperature is $y'(t)$. Since this rate of change is given to be proportional to the difference between the potato's temperature and that of the surrounding room (i.e. $y(t) - 65$), the resulting differential equation is $y'(t) = -k(y(t) - 65)$. Note that k is a positive constant since $y'(t)$ must be negative (i.e. the potato is cooling) and since $y(t) - 65 > 0$ (i.e. the potato is hotter than the surrounding room).

8. Let $x(t)$ be the position (displacement) of the particle at time t. The force on the particle is given to be proportional to this displacement. Therefore, the force, F, is equal to $-kx(t)$ where k is a positive constant. The negative sign is present since the direction of F is opposite to that of $x(t)$. Newton's law states $F = ma$ where m is the mass of the object and $a = x''(t)$ is its acceleration. Therefore, $F = ma$ becomes $-kx(t) = mx''(t)$, which is the differential equation governing the motion of this particle.

9. Let $x(t)$ be the position (displacement) of the particle at time t. The force on the particle is given to be proportional to the square of the particle's velocity, i.e. $(x'(t))^2$. As a first guess, one might surmise that the force is given by $F = -k(x'(t))^2$, where k is a positive constant. However, closer inspection reveals that this will have the force pointing to the left, regardless of whether the velocity is positive or negative. We can work around this difficulty by letting the force equal $F = -kx'(t)|x'(t)|$. The reader will recognize that the force is positive when $x'(t) < 0$, while the force is negative when $x'(t) > 0$, thus insuring that the force is always opposite the particle's motion. Newton's law states $F = ma$ where m is the mass of the object and $a = x''(t)$ is its acceleration. Therefore, $F = ma$ becomes $-k(x'(t))|x'(t)| = mx''(t)$, which is the differential equation governing the motion of this particle.

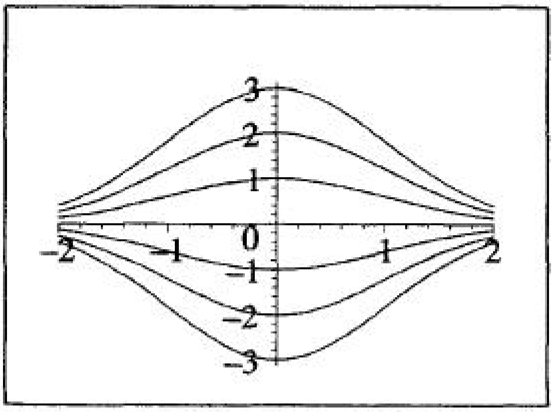
10. Let $x(t)$ be the position (displacement) of the particle at time t. The force on the particle is given to be inversely proportional to the square of this displacement. The direction of F is opposite to that of $x(t)$. Therefore, the force, F, is equal to $-k/[x(t)|x(t)|]$ where k is a positive constant. Note that we have written $x(t)|x(t)|$ instead of $x(t)^2$ since $-k/[x(t)|x(t)|]$ is negative when $x(t)$ is positive and $-k/[x(t)|x(t)|]$ is positive when $x(t)$ is negative. This agrees with the desired direction of F. Newton's law states $F = ma$ where m is the mass of the object and $a = x''(t)$ is its acceleration. Therefore, $F = ma$ becomes

$$\frac{-k}{x(t)|x(t)|} = mx''(t)$$

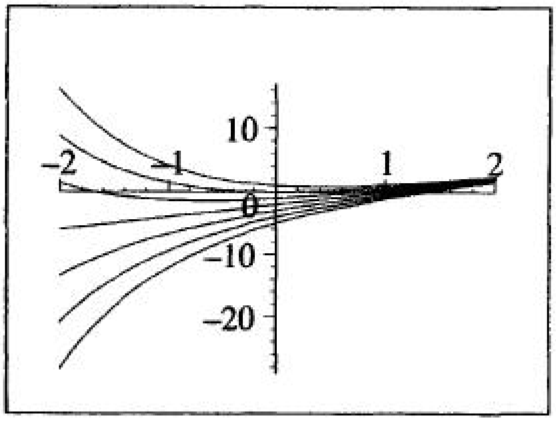
which is the differential equation governing the motion of this particle.

Section 2.1

3. $y'(t) = -Cte^{-(1/2)t^2}$ and $-ty(t) = -tCe^{-(1/2)t^2}$, so $y' = -ty$.



4. $y'(t) + y(t) = (2 - Ce^{-t}) + (2t - 2 + Ce^{-t}) = 2t$

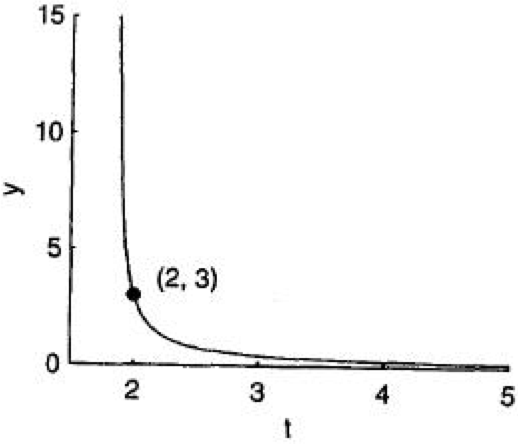


6. If $y(t) = 4/(1 + Ce^{-4t})$, then

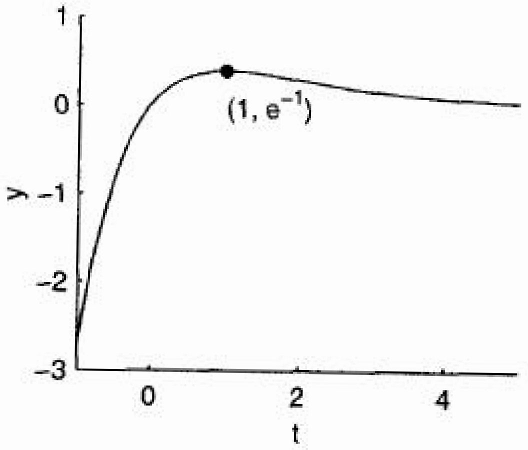
$$y' = \frac{16C\,e^{(-4t)}}{(1 + C\,e^{(-4t)})^2}$$

$$\begin{aligned} y(4 - y) &= \frac{4}{1 + Ce^{-4t}} \times \left[4 - \frac{4}{1 + Ce^{-4t}}\right] \\ &= \frac{16(1 + Ce^{-4t}) - 16}{(1 + C\,e^{(-4t)})^2} \\ &= \frac{16C\,e^{(-4t)}}{(1 + C\,e^{(-4t)})^2} \end{aligned}$$

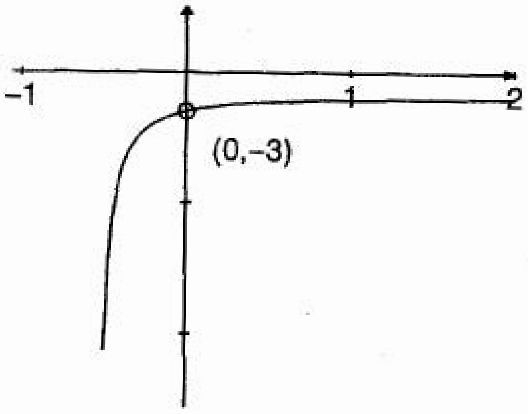
12. $y(t) = (4/17)\cos t + (1/17)\sin t - (21/17)e^{-4t}$ on interval $(11/6, \infty)$.



14. We need $e^{-1} = y(1) = e^{-1}(1 + C/(1)) = (1 + C)e^{-1}$. Hence $C = 0$, and our solution is $y(t) = te^{-t}$. This function is defined and differentiable on the whole real line. Hence the interval of existence is the whole real line.



15. $y(t) = 2/(-1+e^{-2t/3})$. The interval of existence is $(-\ln(3)/2, \infty)$.



Section 2.2

1. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= xy\\ \frac{dy}{y} &= x\,dx\\ \ln|y| &= \frac{1}{2}x^2 + C\\ |y(x)| &= e^{x^2/2+C}\\ y(x) &= \pm e^C \cdot e^{x^2/2}\\ &= Ae^{x^2/2},\end{aligned}$$

Where the constant $A = \pm e^C$ is arbitrary.

2. Separate the variables and integrate.

$$\begin{aligned}x\frac{dy}{dx} &= 2y\\ \frac{1}{y}dy &= \frac{2}{x}dx\\ \ln|y| &= 2\ln|x| + C\\ |y| &= e^{\ln x^2+C}\\ y(x) &= \pm e^C x^2\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = Dx^2$.

6. Separate the variables and integrate. *Note: Factor the right-hand side.*

$$\begin{aligned}\frac{dy}{dx} &= (e^x + 1)(y - 2)\\ \frac{1}{y-2}\,dy &= (e^x + 1)\,dx\\ \ln|y - 2| &= e^x + x + C\\ |y - 2| &= e^{e^x+x+C}\\ y - 2 &= \pm e^C e^{e^x+x}\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = De^{e^x+x} + 2$.

8. Separate the variables and integrate.

$$\begin{aligned}
\frac{dy}{dx} &= y\left(\frac{x}{x-1}\right)\\
\frac{1}{y}\,dy &= \left(1+\frac{1}{x-1}\right)dx\\
\ln|y| &= x+\ln|x-1|+C\\
|y| &= e^{x+\ln|x-1|+C}\\
y(x) &= \pm e^{C}e^{x}e^{\ln|x-1|}
\end{aligned}$$

Letting $D = \pm e^{C}$, $y(x) = De^{x}|x-1|$. It is important to note that this solution is not differentiable at $x = 1$ and further information (perhaps in the form of an initial condition) is needed to remove the absolute value and determine the interval of existence.

13.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x}\\
\frac{dy}{y} &= \frac{dx}{x}\\
\lambda|y| &= \ln|x|+C\\
|y(x)| &= e^{\ln|x|+C} = e^{C}\,|x|\\
y(x) &= Ax.
\end{aligned}$$

The initial condition $y(1) = -2$ gives $A = -2$. The solution is $y(x) = -2x$. The solution is defined for all x, but the differential equation is not defined at $x = 0$ so the interval of existence is $(0, \infty)$.

14.

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{2t(1+y^2)}{y}\\
\frac{y\,dy}{1+y^2} &= -2t\,dt\\
\frac{1}{2}\ln(1+y^2) &= -t^2+C\\
1+y^2 &= e^{-2t^2+C} = e^{C}e^{-2t^2}\\
1+y^2 &= Ae^{-2t^2}
\end{aligned}$$

With $y(0) = 1$, $1+1^2 = Ae^{-2(0)^2}$ and $A = 2$. Thus,

$$\begin{aligned}
1+y^2 &= 2e^{-2t^2}\\
y &= \pm\sqrt{2e^{-2t^2}-1}.
\end{aligned}$$

We must choose the branch that contains the initial condition $y(0) = 1$. Thus, $y = \sqrt{2e^{-2t^2}-1}$. This solution is defined, provided that

$$\begin{aligned}
2e^{-2t^2}-1 &> 0\\
e^{-2t^2} &> \frac{1}{2}\\
-2t^2 &> \ln\frac{1}{2}\\
t^2 &< -2\ln\frac{1}{2}\\
t^2 &< \ln 4\\
|t| &< \sqrt{\ln 4}.
\end{aligned}$$

Thus, the interval of existence is $(-\sqrt{\ln 4}, \sqrt{\ln 4})$.

15.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin x}{y}\\
y\,dy &= \sin x\,dx\\
\frac{1}{2}y^2 &= -\cos x + C_1\\
y^2 &= -2\cos x + C \quad (C = 2C_1)\\
y(x) &= \pm\sqrt{C-2\cos x}
\end{aligned}$$

Using the initial condition we notice that we need the plus sign, and $1 = y(\pi/2) = \sqrt{C}$. Thus $C = 1$ and the solution is

$$y(x) = \sqrt{1-2\cos x}.$$

The interval of existence will be the interval containing $\pi/2$ where $2\cos x < 1$. This is $\pi/3 < x < 5\pi/3$.

16.

$$\begin{aligned}
\frac{dy}{dx} &= e^{x+y} \\
e^{-y}\,dy &= e^x\,dx \\
-e^{-y} &= e^x + C \\
e^{-y} &= -e^x - C \\
-y &= \ln(-e^x - C) \\
y &= -\ln(-e^x - C)
\end{aligned}$$

With $y(0) = 1$,

$$\begin{aligned}
1 &= -\ln(-e^0 - C) \\
-1 - C &= e^{-1} \\
C &= -1 - e^{-1}.
\end{aligned}$$

Thus,

$$y = -\ln(-e^x + e^{-1} + 1).$$

This solution is defined provided that

$$\begin{aligned}
-e^x + e^{-1} + 1 &> 0 \\
e^x &< e^{-1} + 1 \\
x &< \ln(e^{-1} + 1).
\end{aligned}$$

Thus, the interval of existence is $(-\infty, \ln(e^{-1} + 1))$.

34. Let $y(t)$ be the temperature of the beer at time t minutes after being placed into the room. From Newton's law of cooling, we obtain

$$y'(t) = k(70 - y(t)) \quad y(0) = 40$$

Note k is positive since $70 > y(t)$ and $y'(t) > 0$ (the beer is warming up). This equation separates as

$$\frac{dy}{70 - y} = k\,dt$$

which has solution $y = 70 - Ce^{-kt}$. From the initial condition, $y(0) = 40$, $C = 30$. Using $y(10) = 48$, we obtain $48 = 70 - 30e^{-10k}$ or $k = (-1/10)\ln(11/15)$ or $k = .0310$. When $t = 25$, we obtain $y(25) = 70 - 30e^{-.598} \approx 56.18°$.

35. The same differential equation and solution hold as in the previous problem:

$$y(t) = 70 - Ce^{-kt}$$

We let $t = 0$ correspond to when the beer was discovered, so $y(0) = 50$. This means $C = 20$. We also have $y(10) = 60$ or

$$60 = 70 - 20e^{-10k}$$

Therefore, $k = (-1/10)\ln(1/2) \approx .0693$. We want to find the time T when $y(T) = 40$, which gives the equation

$$70 - 20e^{-kT} = 40$$

Since we know k, we can solve this equation for T to obtain

$$T = (-1/k)\ln(3/2) \approx -5.85$$

or about 5.85 minutes before the beer was discovered on the counter.