Section 4.7

9. (a) The displacement satisfies the differential equation $x'' + 4x = 4\cos\omega t$, with initial conditions $x(0) = x'(0) = 0$. The solution is

$$x(t) = \frac{4}{4-\omega^2}(\cos\omega t - \cos 2t).$$

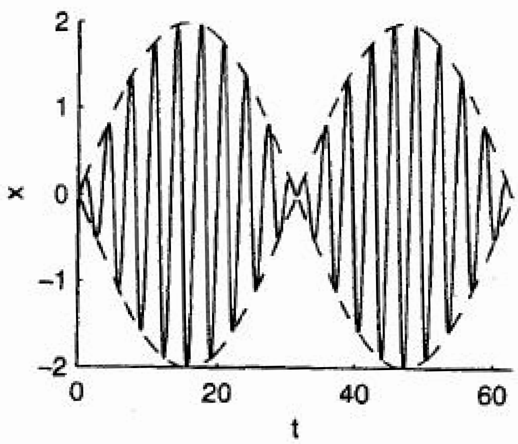
(b) If we set $\overline{\omega} = (2+\omega)/2$, and $\delta = (2-\omega)/2$, the solution becomes

$$x(t) = \frac{2}{\overline{\omega}\delta}\sin\delta t \sin\overline{\omega} t.$$

We will take $\omega = 1.8$, which is near to $\omega_0 = 2$. Then $\overline{\omega} = 1.9$, and $\delta = 0.1$, so

$$x(t) = \frac{2}{0.19}\sin 0.1t \sin 1.9t.$$

The graph of x and its envelope is presented in the following figure.



10. (a) As in the text we find the particular solution

$$x_p(t) = \frac{A}{2\omega_0} t \sin\omega_0 t = \frac{2}{5} t \sin 5t.$$

The solution of the homogeneous equation is $x_h(t) = C_1\cos 5t + C_2\sin 5t$, so our solution has the form

$$x(t) = C_1\cos 5t + C_2\sin 5t + \frac{2}{5} t \sin 5t.$$

Our initial conditions are

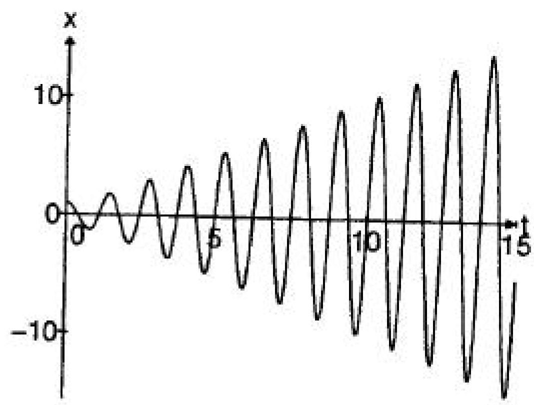
$$\begin{aligned} 1 &= x(0) = C_1 \\ 0 &= x'(0) = 5C_2, \end{aligned}$$

so the solution is

$$x(t) = \cos 5t + \frac{2}{5} t \sin 5t.$$

The particular solution $x_p(t)$ has a factor of t so its amplitude will grow, indicating a resonant solution.

(b)



17. We will use the complex method, and look for a solution of the equation $z'' + 7z' + 10z = 3e^{3it}$ of the form $z(t) = ae^{3it}$. Then our particular solution will be $x_p = \text{Re}\, z$. Differentiating we get

$$\begin{aligned} z'' + 7z' + 10z &= a((3i)^2 + 7(3i) + 10)e^{3it} \\ &= a(1 + 21i)e^{3it} = 3e^{3it}. \end{aligned}$$

Hence

$$a = \frac{3}{1 + 21i} = 3 \cdot \frac{1 - 21i}{442},$$

so

$$\begin{aligned} z(t) &= 3 \cdot \frac{1 - 21i}{442} e^{3it} \\ &= \frac{3}{442}[1 - 21i][\cos 3t + i \sin 3t] \\ &= \frac{3}{442}[(\cos 3t + 21 \sin 3t) \\ &\quad + i(\sin 3t - 21 \cos 3t)], \end{aligned}$$

and $x_p(t) = \text{Re}\, z(t) = 3(\cos 3t + 21 \sin 3t)/442$. The characteristic polynomial is $P(\lambda) = \lambda^2 + 7\lambda + 10 = (\lambda + 2)(\lambda + 5)$, which has roots -2 and -5. Hence the general solution to the homogenous equation is $x_h(t) = C_1 e^{-2t} + C_2 e^{-5t}$. The general solution to the inhomogeneous equation is

$$x(t) = \frac{3}{442}(\cos 3t + 21 \sin 3t) + C_1 e^{-2t} + C_2 e^{-5t}.$$

The initial conditions imply that

$$\begin{aligned} -1 = x(0) &= \frac{3}{442} + C_1 + C_2 \\ 0 = x'(0) &= \frac{189}{442} - 2C_1 - 5C_2, \end{aligned}$$

or

$$\begin{aligned} C_1 + C_2 &= -\frac{445}{442} \\ 2C_1 + 5C_2 &= \frac{189}{442}. \end{aligned}$$

The solutions are $C_1 = -71/39$ and $C_2 = 83/102$, so the solution to the initial value problem is

$$x(t) = \frac{3}{442}(\cos 3t + 21 \sin 3t) - \frac{71}{39}e^{-2t} + \frac{83}{102}e^{-5t}.$$

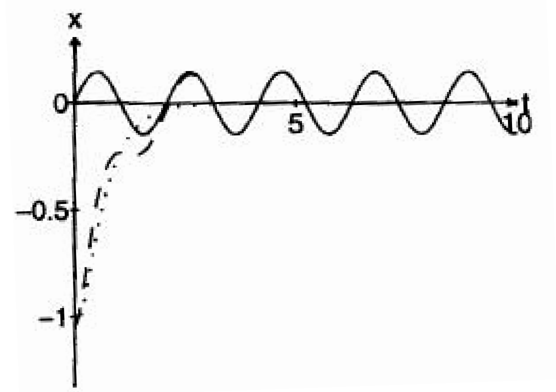
The steady-state solution is the particular solution

$$x_p(t) = \frac{3}{442}(\cos 3t + 21 \sin 3t),$$

and the transient response is

$$x_h(t) = -\frac{71}{39}e^{-2t} + \frac{83}{102}e^{-5t}.$$

In the following plot the graph of the solution to the initial value problem is the dashed curve, the transient response is dotted, and the steady-state solution is solid.



Section 8.1

10. If $x(t) = e^t$ and $y(t) = e^{-t}$, then

$$x' = (e^t)' = e^t,$$

and

$$x^2 y = (e^t)^2 (e^{-t}) = e^t,$$

and the first equation is satisfied. Further,

$$y' = (e^{-t})' = -e^{-t},$$

and

$$-xy^2 = -e^{-t}(e^{-t})^2 = -e^{-t},$$

so the second equation is satisfied. Finally, $x(0) = e^0 = 1$ and $y(0) = e^{-0} = 1$, so the initial conditions are satisfied.

11. We know that $y'' = -2y' - 4y + 3\cos 2t$, $y(0) = 1$, $y'(0) = 0$. If we let $u_1 = y$ and $u_2 = y'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -2u_2 - 4u_1 + 3\cos 2t. \end{aligned}$$

Hence with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -2u_2 - 4u_1 + 3\cos 2t \end{pmatrix}.$$

Furthermore $\mathbf{u}(0) = (u_1(0), u_2(0))^T = (y(0), y'(0))^T = (1, 0)^T$.

16. We know that $y''' = -y'y'' + \sin \omega t$, $y(0) = \alpha$, $y'(0) = \beta$, and $y''(0) = \gamma$. If we let $u_1 = y$, $u_2 = y'$, and $u_3 = y''$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= -u_2 u_3 + \sin \omega t. \end{aligned}$$

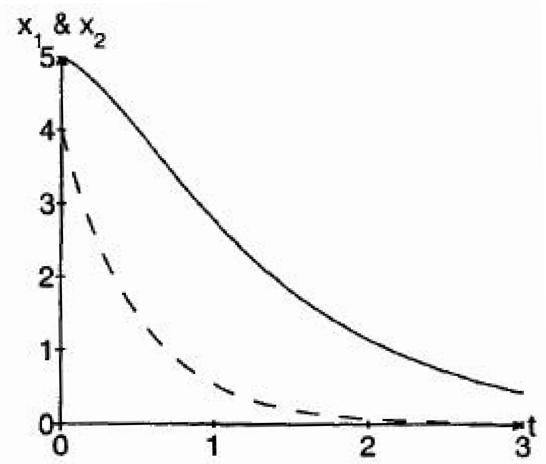
With $\mathbf{u} = (u_1, u_2, u_3)^T$,

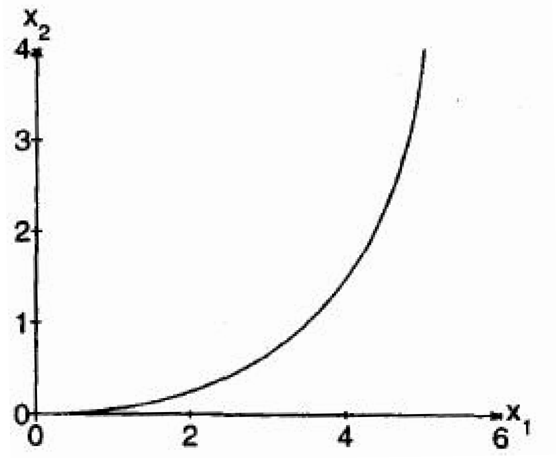
$$\mathbf{u}' = \begin{pmatrix} u_2 \\ u_3 \\ -u_2 u_3 + \sin \omega t \end{pmatrix}.$$

Furthermore, $\vec{u}(0) = (u_1(0), u_2(0), u_3(0))^T = (y(0), y'(0), y''(0))^T = (\alpha, \beta, \gamma)^T$.

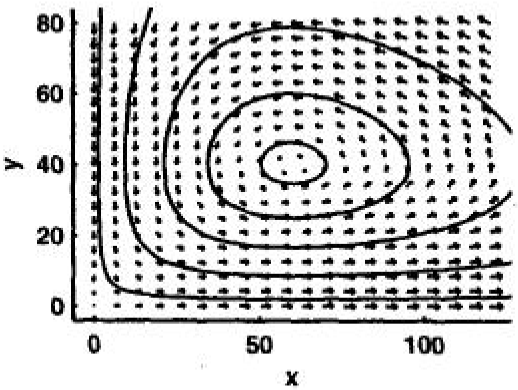
Section 8.2

2. The first figure is a plot of $x_1(t) = 9e^{-t} - 4e^{-2t}$ (the solid curve) and $x_2(t) = 4e^{-2t}$ (the dashed curve) versus t on the time interval $[0, 3]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.

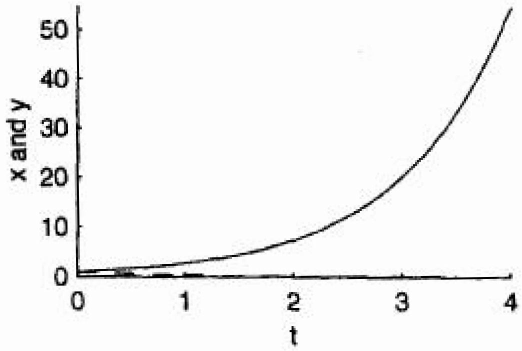


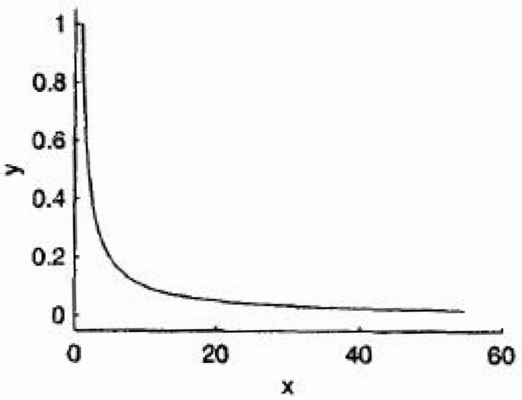


15. The directions and several solution trajectories for $x' = (0.4 - 0.01y)x$ and $y' = (0.005x - 0.3)y$ follow.

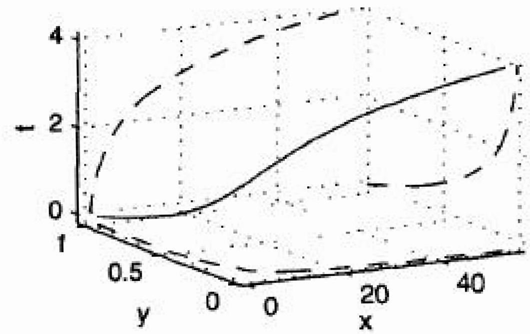


18. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.





Finally, a composite plot, with the 3D plot solid and the others dashed.



Section 8.3

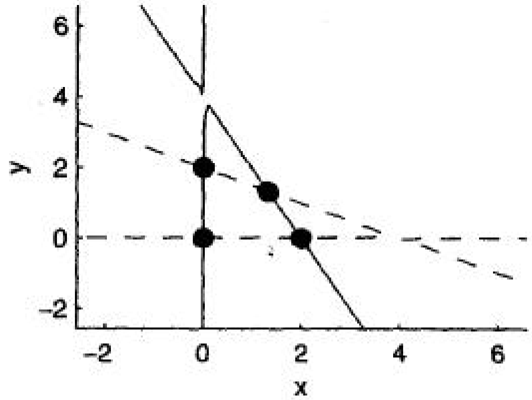
2. Set the right-hand side of $x' = 4x - 2x^2 - xy$ equal to zero.

$$4x - 2x^2 - xy = 0$$
$$x(4 - 2x - y) = 0$$

Thus, $x = 0$ and $4 - 2x - y = 0$ are the x-nullclines. They appear in a solid line style in the figure. Set the right-hand side of $y' = 4y - xy - 2y^2$ equal to zero.

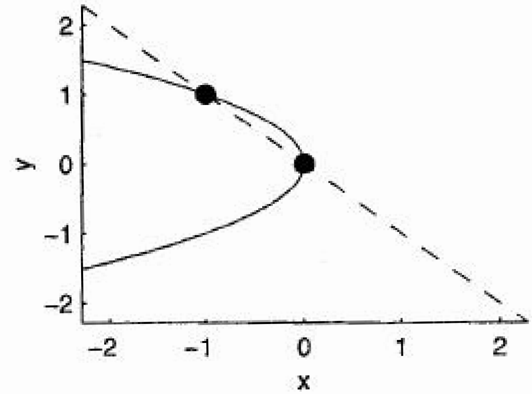
$$4y - xy - 2y^2 = 0$$
$$y(4 - x - 2y) = 0$$

Thus, $y = 0$ and $4 - x - 2y = 0$ are the y-nullclines. They appear in a dashed line style in the figure.



The equilibrium points appear where the x-nullclines intersect the y-nullclines. These are $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(4/3, 4/3)$.

6. Set the right-hand side of $x' = x + y^2$ equal to zero. Thus, $x + y^2 = 0$ is the x-nullcline, set in a solid line style in the figure. Next, set the right-hand side of $y' = x + y$ equal to zero. Thus, $x + y = 0$ is the y-nullcline, set in a dashed line style in the figure.



The equilibrium points appear where the x-nullclines intersect the y-nullclines. These are at $(0, 0)$ and $(-1, 1)$.

7. (a) If $x(t) = t$ and $y(t) = \sin t$, then

$$x' = (t)' = 1,$$

and

$$1-(y-\sin x)\cos x = 1-(\sin t-\sin t)\cos t = 1,$$

so the first equation is satisfied. Further,

$$y' = (\sin t)' = \cos t,$$

and

$$\cos x - y + \sin x = \cos t - \sin t + \sin t = \cos t,$$

so the second equation is satisfied.

(b) See the figure in part (c).

(c) Because of uniqueness, the solution with initial condition $x(0) = \pi/2$, $y(0) = 0$, cannot cross the solution $x = t$, $y = \sin t$ found in part (a). Thus, it must remain below the solution in part (a) for all time. Therefore, if $(x(t), y(t))$ denotes the second solution, we must have $y(t) < \sin x(t)$ for all time, as shown in the figure.