Section 6.1

3. We begin with $t_0 = 0$, $y_0 = 1$ and $f(t, y) = ty$. First, the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 0 \times 1 = 0$$
$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1) = 0.1 \times 1 = 0.1$$

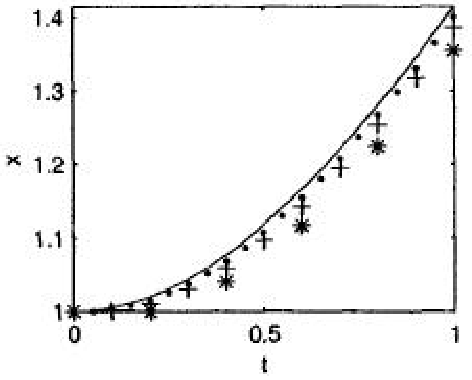
Update t and y.

$$y_1 = y_0 + h\frac{s_1 + s_2}{2} = 1 + 0.1\frac{0 + 0.1}{2} = 1.005$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	t_k	y_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	1.0000	0.0000	0.1000	0.1	0.0050
1	0.1	1.0050	0.1005	0.2030	0.1	0.0152
2	0.2	1.0202	0.2040	0.3122	0.1	0.0258
3	0.3	1.0460	0.3138	0.4309	0.1	0.0372
4	0.4	1.0832	0.4333	0.5633	0.1	0.0498
5	0.5	1.1331	0.5665	0.7138	0.1	0.0640

9 The initial condition $x(0) = 1$ provides $C = 1/2$ and the implicit solution $x^2 = t^2 + 1$. Of course, because we want $x(0) = 1$, we take the right-hand branch, $x = \sqrt{t^2 + 1}$. In the image that follows, the exact solution is plotted on the interval $[0, 1]$. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.

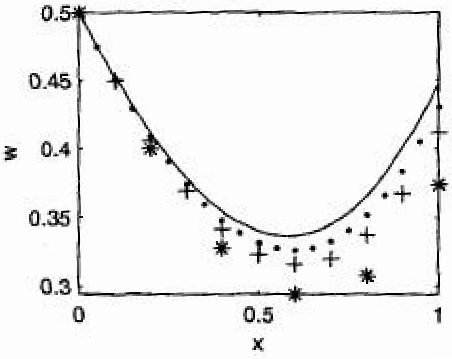


An integrating factor is e^x.

$$e^x w' + e^x w = x^2 e^x$$
$$(e^x w)' = x^2 e^x$$

Integrate by parts.

$$e^x w = x^2 e^x - 2xe^x + 2e^x + C$$
$$w = x^2 - 2x + 2 + Ce^{-x}$$



The initial condition $w(0) = 1/2$ provides $C = -3/2$ and the exact solution $w = x^2 - 2x + 2 - (3/2)e^{-x}$. In the image that follows, the exact solution is plotted on the interval $[0, 1]$. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.

Section 4.1

1. Compare

$$y'' + 3y' + 5y = 3\cos 2t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = 3$, $q(t) = 5$, and $g(t) = 3\cos 2t$. Hence, the equation is linear and inhomogeneous.

2. Divide both sides of $t^2y'' = 4y' - \sin t$ by t^2, then rearrange to obtain

$$y'' - \frac{4}{t^2}y' = -\frac{\sin t}{t^2}.$$

Compare this with

7. In

$$y'' + 3y' + 4\sin y = 0$$

note that the term $4\sin y$ is nonlinear. Hence, this equation is nonlinear.

10. Use $k = (mg)/x$ to determine the spring constant.

$$k = \frac{5\,\text{kg} \times 9.8\,\text{m/s}^2}{0.75\,\text{m}} = 65.3\,\text{N/m}.$$

Using the model

$$my'' + \mu y' + ky = F(t),$$

we note that there is no damping ($\mu = 0$) and there is no driving force ($F(t) = 0$), so the equation becomes

$$my'' + ky = 0.$$

With $m = 5$ kg and $k = 65.3$ N/m, this becomes

$$5y'' + 65.3y = 0.$$

Because the initial displacement was 36 cm upward, $y(0) = -0.36$ (assuming an orientation where y is positive in the downward direction). Because the mass is given an initial downward velocity of 0.45 m/s, $y'(0) = 0.45$. Because we've chosen the downward direction to represent positive y-values, downward velocities are positive.

12. The period of the driving force is 4 seconds. Thus, the circular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}\ \text{rad/s}.$$

Because the amplitude is $A = 0.25$ m, and the spring is initially displaced 0.25 m downward (remember, upward is negative) by the driving force, the driving force can be described with

$$F(t) = 0.25\cos\frac{\pi t}{2}$$

Now, $m = 5$ kg, $k = 65.3$ N/m, and the damping force is given by $R(v) = -0.125v$. This makes the damping constant $\mu = 0.125$. Thus, the equation

$$my'' + \mu y' + ky = F(t)$$

becomes

$$5y'' + 0.125y' + 65.3y = 0.25\cos\frac{\pi t}{2}$$

From Exercise 10, the initial conditions are $y(0) = -0.36$ and $y'(0) = 0.45$.

17. We leave it to our readers to first check that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are solutions of $y'' - y' - 2y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t}}{e^{2t}} = e^{-3t} \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^{t},$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

18. We leave it to our readers to first check that $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$ are solutions of $y'' + 9y = 0$. Next note that

$$\frac{y_1(t)}{y_2(t)} = \frac{\cos 3t}{\sin 3t} = \cot 3t \neq c,$$

where c is some constant. Therefore $y_1(t)$ is not a constant multiple of $y_2(t)$ and the solutions are linearly independent. Further,

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$
$$= 3\cos^2 3t + 3\sin^2 3t = 3,$$

which is never equal to zero. Therefore, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent and form a fundamental set of solutions.

Section 4.3

3. Let $y = e^{\lambda t}$ in $y'' + 5y' + 6y = 0$ to obtain

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} = 0,$$
$$e^{\lambda t}(\lambda^2 + 5\lambda + 6) = 0.$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0,$$
$$(\lambda + 3)(\lambda + 2) = 0,$$

and roots $\lambda = -3$ and $\lambda = -2$. Because the roots are distinct, the solution $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-2t}$ form a fundamental set of solutions and the general solution is

10. If $y'' + 4y = 0$, then the characteristic equation is

$$\lambda^2 + 4 = 0.$$

The roots of the characteristic equation are $\pm 2i$, leading to the complex solutions

$$z(t) = e^{2it} \quad \text{and} \quad \overline{z}(t) = e^{-2it}.$$

However, by Euler's identity,

$$z(t) = \cos 2t + i \sin 2t,$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$. Hence, the general solution is

$$y(t) = C_1 \cos 2t + C_2 \sin 2t.$$

14. If $y'' + 2y' + 3y = 0$, then the characteristic equation is

$$\lambda^2 + 2\lambda + 3 = 0.$$

The roots of the characteristic equation are $-1 \pm \sqrt{2}i$, leading to the complex solutions

$$z(t) = e^{(-1+\sqrt{2}i)t} \quad \text{and} \quad \overline{z}(t) = e^{(-1-\sqrt{2}i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t}e^{\sqrt{2}it} = e^{-t}(\cos \sqrt{2}t + i \sin \sqrt{2}t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = e^{-t} \cos \sqrt{2}t$ and $y_2(t) = e^{-t} \sin \sqrt{2}t$. Hence the general solution is

$$y(t) = C_1 e^{-t} \cos \sqrt{2}t + C_2 e^{-t} \sin \sqrt{2}t.$$

21. If $16y'' + 8y' + y = 0$, then the characteristic equation is

$$16\lambda^2 + 8\lambda + 1 = (4\lambda + 1)^2 = 0.$$

Hence, the characteristic equation has a single, double root, $\lambda = -1/4$. Therefore, $y_1(t) = e^{-(1/4)t}$ and $y_2(t) = te^{-(1/4)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-t/4} + C_2 t e^{-t/4} = (C_1 + C_2 t)e^{-t/4}.$$

25. If $y'' - y' - 2y = 0$, then the characteristic equation is

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

with roots $\lambda = 2$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1 e^{2t} + C_2 e^{-t}.$$

Using the initial condition $y(0) = -1$ provides

$$-1 = C_1 + C_2.$$

Differentiating the general solution,

$$y'(t) = 2C_1 e^{2t} - C_2 e^{-t},$$

then using the initial condition $y'(0) = 2$ leads to

$$2 = 2C_1 - C_2.$$

These equations yield $C_1 = 1/3$ and $C_2 = -4/3$, giving the particular solution

$$y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}.$$

29. If $y'' + 10y' + 25y = 0$, then the characteristic equation is

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0,$$

with repeated root $\lambda = -5$. This leads to the fundamental set of solutions $y_1(t) = e^{-5t}$ and $y_2(t) = te^{-5t}$ and the general solution is

$$y(t) = C_1 e^{-5t} + C_2 t e^{-5t} = (C_1 + C_2 t)e^{-5t}.$$

Using the initial condition $y(0) = 2$ leads to

$$2 = C_1.$$

Differentiating the general solution,

$$y'(t) = C_2 e^{-5t} - 5(C_1 + C_2 t)e^{-5t},$$

then the initial condition $y'(0) = -1$ leads to

$$-1 = C_2 - 5C_1.$$

Thus, $C_1 = 2$ and $C_2 = 9$ and the final solutions is

$$y(t) = (2 + 9t)e^{-5t}.$$

34. If $4y'' + y = 0$, then the characteristic equation is $4\lambda^2 + 1 = 0$ with roots $\pm i/2$. The complex solution

$$z(t) = e^{(i/2)t} = \cos\frac{t}{2} + i\sin\frac{t}{2}$$

provides a fundamental set of real solutions, $y_1(t) = \cos(t/2)$ and $y_2(t) = \sin(t/2)$, leading to the general solution

$$y(t) = C_1\cos\frac{t}{2} + C_2\sin\frac{t}{2}.$$

The initial condition $y(1) = 0$ provides

$$0 = C_1\cos\frac{1}{2} + C_2\sin\frac{1}{2}.$$

Differentiating the general solution,

$$y'(t) = -\frac{1}{2}C_1\sin\frac{t}{2} + \frac{1}{2}C_2\cos\frac{t}{2}.$$

The initial condition $y'(1) = -2$ provides

$$-2 = -\frac{1}{2}C_1\sin\frac{1}{2} + \frac{1}{2}C_2\cos\frac{1}{2}.$$

Thus, $C_1 = 4\sin(1/2)$ and $C_2 = -4\cos(1/2)$, and the final solution is

$$\begin{aligned} y(t) &= 4\sin\frac{1}{2}\cos\frac{t}{2} - 4\cos\frac{1}{2}\sin\frac{t}{2}, \\ &= 4\sin\left(\frac{1}{2} - \frac{t}{2}\right), \\ &= -4\sin\frac{1}{2}(t-1). \end{aligned}$$