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1. van der Pol equation for an oscillator with nonlinear damping:

$$x'' - (1 - x^2)x' + x = 0,$$

where x = voltage, y =current; derived in the development of radio and vacuum tubes (along with other related equations, such as the Duffing equation and the general class of Liénard equations)

2. pendulum

$$x'' + bx' + \sin x = 0,$$

where $b > 0$ is the damping coefficient

3. Brusselator, a chemical oscillator:

$$\begin{aligned}x' &= c - (d + 1)x + x^2y, \\y' &= dx - x^2y,\end{aligned}$$

where x, y are concentrations of species in a reaction and c, d are parameters

4. a two-mode laser produces two types of photons, in quantities n_1, n_2 , respectively; using n'_i = gain - loss yields

$$\begin{aligned}n'_1 &= G_1 N n_1 - k_1 n_1, \\n'_2 &= G_2 N n_2 - k_2 n_2,\end{aligned}$$

where $N = N_0 - a_1 n_1 - a_2 n_2$ is the number of excited atoms in the laser and all parameters are positive

5. a relativistic equation for the orbit of a planet around the sun is

$$\begin{aligned}u' &= v, \\v' &= a - u + \epsilon u^2,\end{aligned}$$

where $\iota = d/d\theta$ and $u = 1/r$ for (r, θ) representing the planet's position in polar coordinates with the sun at the origin and where $a > 0$ and $0 < \epsilon \ll 1$ (the Einstein relativistic correction)

6. a glider moving with speed v and angle θ relative to the horizon satisfies

$$\begin{aligned}v' &= -\sin \theta - Dv^2, \\v\theta' &= -\cos \theta + v^2,\end{aligned}$$

where Dv^2 is a drag term and v^2 is a lift term

7. if two bar magnets are pinned at a common point midway along their lengths with their ends free to rotate in a plane about this point, then their angles θ_1, θ_2 with the horizontal satisfy

$$\theta_1' = K \sin(\theta_1 - \theta_2) - \sin(\theta_1),$$

$$\theta_2' = K \sin(\theta_2 - \theta_1) - \sin(\theta_2),$$

where $K \geq 0$ is a parameter

8. Lorenz equations for atmospheric turbulence are sometimes reduced to

$$x' = -ax + ay,$$

$$y' = rx - y - xz,$$

$$z' = -bz + xy,$$

where $a, b, r > 0$ are parameters; this has an equilibrium point at $(0, 0, 0)$ for all r and 2 other equilibria if $r > 1$; the origin switches from asymptotically stable to unstable at $r = 1$, while the other equilibria lost stability at $r = 470/19$ and chaos results!

See S. Strogatz's book "Nonlinear Dynamics and Chaos" for more examples and a lot more on phase portraits!