Is Trivial Dynamics That Trivial?

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1. INTRODUCTION. Arguably dynamical systems is the branch of mathematics that has developed most rapidly in the last quarter of a century. Some facts supporting this statement: since 1980, no less than thirty new journals have appeared that are either specifically or primarily devoted to the subject; the 2000 revision of the Mathematical Subject Classification included dynamical systems (together with ergodic theory) as the separate subject 37-xx; Jean-Christophe Yoccoz (1994) and Curtis T. McMullen (1998) have recently been awarded the prestigious Fields Medal for their contributions to the field.

One-dimensional discrete dynamics have a particularly appealing feature: with a minimal amount of technical background, available even at the undergraduate level (well, not quite so; more on this later), one can quickly become acquainted with the essential paradigms that pervade the whole theory of dynamical systems. In tune with the times, this MONTHLY has devoted a comparatively large number of papers to this particular item in the last few years [1]–[3], [9]–[11], [16], [21], [36], [38], [41], [46], [47]. When referring to such a dynamical system, we are speaking about a continuous map \( f \) from the unit interval \( I = [0, 1] \) into itself, and we are interested in studying the asymptotic behavior of the orbits \( \left( f^n(x) \right)_{n=0}^{\infty} \) for \( x \) in \( I \), where \( f^0 \) is the identity map and \( f^{n+1} = f \circ f^n \) if \( n \geq 0 \).

It is little wonder then that two papers on intervals dynamics, Sharkovsky’s [42] and Li and Yorke’s [23], played pivotal roles in the upsurge of dynamical systems in the mid-seventies. In fact, if an imaginary contest for the most cited paper ever in the history of dynamics were held, both papers would be hot contenders for the gold medal. The results in the papers will be relevant to us later, so a brief technical interlude is in order to explain them.

In the Sharkovsky ordering \( > \), the set of positive integers (together with the additional symbol \( "2\infty" \)) is ordered as follows:

\[
3 > 5 > 7 > \cdots > 2 \cdot 3 > 2 \cdot 5 > 2 \cdot 7 > \cdots > 4 \cdot 3 > 4 \cdot 5 > 4 \cdot 7 \cdots > \cdots
\]

\[
> 2^n \cdot 3 > 2^n \cdot 5 > 2^n \cdot 7 > \cdots > \cdots > 2\infty > \cdots > 2^n > \cdots > 4 > 2 > 1
\]

Thus, in this order odd numbers (except 1) are the largest ones and are written in an order opposite to their usual one; in particular, 3 is the largest number of all. Next follow the numbers \( 2 \cdot (2s + 1) \) for all positive integers \( s \) (again in the reverse of their standard order), then the numbers \( 2^2 \cdot (2s + 1) \), the numbers \( 2^3 \cdot (2s + 1) \), and so on. In this way we exhaust all numbers except the powers of two, which are ordered (following the intermediate symbol \( 2\infty \)) in the standard way. Thus, as usual, 1 is the smallest positive integer. For instance, we have 15 > 1007 > 222 > 80 > 64.

Recall that a point \( p \) of \( I \) is said to be periodic for a map \( f \) belonging to \( C(I) \) if there is a minimal positive integer \( r \) (the period of \( p \)) such that \( f^r(p) = p \). Sharkovsky’s theorem [42] (see [10] for the simplest available proof) asserts that for any map \( f \) in \( C(I) \) there is some \( t \) in \( \mathbb{Z}^+ \cup \{2\infty\} \) (the type of \( f \)) such that the set of periods corresponding to periodic points of \( f \) is exactly \( \{ s \in \mathbb{Z}^+ : t \geq s \} \). This paper deals mainly with maps of type 1, so we emphasize that these are characterized by the
property that their only periodic points are their fixed points. Two clear-cut examples of type-1 functions are increasing continuous mappings and differentiable maps whose derivatives are smaller than one in absolute value. Namely, if \( f \) is increasing, then the property \( f(x) \geq x \) (respectively, \( f(x) \leq x \)) for an \( x \) in \( I \) implies that the sequence \( \left(f^n(x)\right) \) is increasing (respectively, decreasing), hence a point \( p \) cannot be periodic unless \( f(p) = p \). (By the way, notice that if \( f \) is decreasing then, although \( f^2 \) is increasing, it need not be of type 1: an example is \( f(x) = 1 - x \).

If \( |f'(x)| < 1 \) for each point \( x \) in \( I \), then the contractive mapping theorem applies and all sequences \( \left(f^n(x)\right) \) converge to the same point \( p \), the only fixed point (and the only periodic point) of the map \( f \).

Let \( f \) be a member of \( C(I) \). A set \( S \) is \textit{scrambled} (relative to \( f \)) if for each pair of distinct points \( x \) and \( y \) of \( S \) it is the case that both

\[
\lim_{n \to \infty} \sup |f^n(x) - f^n(y)| > 0
\]

and

\[
\lim_{n \to \infty} \inf |f^n(x) - f^n(y)| = 0.
\]

Li and Yorke proved in [23] that if a map has a periodic point of period three, then it possesses an uncountable scrambled set. Maps possessing such sets are usually described as \textit{chaotic in the sense of Li and Yorke}.

Sharkovsky’s theorem is one of exceptional beauty and has the additional merit of illustrating graphically the way to asymptotic complexity: the larger the type of a map, the more intricate its dynamics. For instance, one could try to show (it is an easy exercise following directly from the Sharkovsky and Li-Yorke theorems) that maps of type greater than \( 2^\infty \) are chaotic in the sense of Li and Yorke; on the other hand, no map of type less than \( 2^\infty \) can be chaotic (we will return to this later). The historical significance of Li and Yorke’s paper lies partly in the fact that it was the first publication in which anybody dared to suggest a plausible and rigorous definition of “chaos.” Remarkably, to get these results no tools are required other than the intermediate value theorem and a lot of ingenuity. We cite [15, p. 43]: “This interesting and beautiful result [Sharkovsky’s theorem] depends only on the continuity of the function in question and should give pause to anyone who might think that all of the really good theorems which don’t rely on the mathematical equivalent of a B-1 bomber were proven before the turn of the [twentieth] century.”

By the way, there is an amusing anecdote linking these works. Li and Yorke’s result is often wrongly quoted (for instance, in [35]) as a mere consequence of Sharkovsky’s theorem, because they also proved that if a map has a periodic point of period three, then it has periodic points of all possible periods. Indeed, Sharkovsky had published his theorem in a Ukrainian journal (in Russian) in the early sixties, but at the time nobody had taken notice of it. According to Gleick [12], Sharkovsky approached Yorke at a conference in East Berlin. There was something very important the Ukrainian had to tell Yorke. In true cold war fashion, a meeting was arranged on a boat on the Spree River. There, with the help of an anonymous Polish translator, Sharkovsky informed Yorke about his discovery (but only sent him the proof several months later).

Yet, to paraphrase the old saying, the future of dynamical systems is not what it used to be. Strange as it may seem, neither Sharkovsky’s theorem nor Li-Yorke’s definition of chaos plays a significant role in what is supposed to be the core of modern one-dimensional dynamics. De Melo and van Strien’s \textit{One-Dimensional Dynamics},
which is presently considered “the book” on this subject,\(^1\) includes a full proof of Sharkovsky’s theorem (see [35, chap. 2]) but earlier advises [35, p. 82]: “Only Sections 2 and 6 are relevant to the remainder of the book; the reader could skip the other sections.” The fate of the Li-Yorke paper is even worse, rating only a tiny (and inaccurate) reference on page 3.

This “modern” approach to dynamics dates back as far as 1976, when in another seminal paper of the seventies, May put an emphasis on the (very complicated) dynamics of the logistic family \(f_\alpha(x) = \alpha x(1 - x)\), with \(0 \leq \alpha \leq 4\), which naturally arises in models from population biology [34]. Then, three years later, Guckenheimer [14] published a remarkably mature paper for the time, in which, among other things, it was proved that if the turning point \(p = 1/2\) is a periodic point of a logistic map \(f_\alpha\), then almost all points (in the sense of Lebesgue measure) are asymptotically periodic and their orbits are attracted to the (finite) orbit of \(p\). The meaning of this is that for almost every \(x\) there is some \(q = q_\delta\) in the orbit of \(p\) such that

\[
\lim_{n \to \infty} |f^n(x) - f^n(q)| = 0.
\]

Guckenheimer’s paper laid the foundations for what could be called the “almost everywhere approach” to dynamics: if a map \(f\) is sufficiently smooth, then there may be a fair chance of providing an effective description of the asymptotic behavior of almost every point of the interval.

Hence a striking difference arises when this measure-theoretic approach is compared with the purely topological one favored by the Sharkovsky and Li-Yorke papers. For instance, in the particular case when \(\alpha = 3.83187\ldots\), the point \(1/2\) is periodic of period three, so \(f_\alpha\) has periodic points of all periods and is chaotic in the Li-Yorke sense. At the same time, almost all points of \(I\) are asymptotically periodic, so any scrambled set \(S\) for \(f_\alpha\) must have measure zero (because \(S\) can include at most one asymptotically periodic point). Thus, while \(S\) is uncountable, the associated chaos is invisible. This state of affairs was neatly, if a bit harshly, described by Collet and Eckmann in a standard (and very influential) monograph of the early eighties [6, pp. 21–22] (capital letters and underlined text are theirs; italics are ours):

**IMPORTANT REMARK:** When we have discussed above the “typical” behavior of a map, we have always insisted on analyzing what happen to most initial points. This is motivated by the fact that we want to make general statements about the behavior of dynamical systems. It happens very often that while most of the initial orbits show a very regular behavior (i.e., they approach a stable periodic orbit), some other initial points—very few in the sense of Lebesgue measure—behave rather in the ergodic way described above. Such a situation has been described in the paper by Li and Yorke “Period three implies chaos”. They show, among other things, that if a map has a (stable or unstable) orbit of period three, then the map in question shows sensitive dependence to initial conditions for an uncountable set of pairs of initial points. But—and maybe the paper did not make this sufficiently clear—for most other points this need not be the case, and hence from a physical point of view the chaotic behavior may be essentially unobservable.

Lately, the area of smooth interval dynamics has undergone another impressive development, not in the least because of the efforts of some truly first-rate mathematicians (names like Blokh, de Melo, Misiurewicz, Lypubich, or van Strien come readily to

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\(^1\)Reviewing it for *Zentralblatt*, G. Świątek writes [44, p. 353]: “For specialists in the area, the book may prove to be an indispensable guide to the literature on the subject. This is due to its clear structure, numerous historical remarks and a careful bibliographical list. I keep it within reach of my hand together with a LaTeX manual and Webster’s dictionary.”
mind). As its crowning achievement (based, of course, on the previous work of many authors) we mention Lyubich’s paper [29], where the “almost every point-almost every parameter” dynamical structure of the logistic family was finally unravelled. Unfortunately, a rather high price had to be paid for the progress. The proofs in [29] and many related papers are extremely involved and often require a battery of sophisticated tools (complex analysis, hyperbolic geometry, etc.). In fact, one of the leading experts in the field recently intimated to the second author that he is unable to understand some of the arguments (according to him, only a score of specialists do!).

The situation resembles that of Fermat’s Last Theorem: results are very easy to state; proofs are extremely difficult to read. The analogy can be carried further. It may have passed generally unnoticed that the proof in the key paper [27] is marred by a substantial gap, for the oversight was found and filled in several years later [30]. As far as we know, the correction has not yet been published. Remarkably enough there is also a gap in Guckenheimer’s paper (see [40, p. 180]). Fortunately, it is rather harmless (it does not affect Guckenheimer’s result that we mentioned earlier and was corrected by Mañé in [31]).

2. MAIN RESULTS. At this point, a reader interested in getting the flavor of this substantial part of interval dynamics but not eager to delve into subtle technicalities will surely think that he or she is in a hopeless situation. We will try to convince him or her that this need not be the case.

Since order and chaos are the yin and yang of modern dynamics, we could very well start by proposing reasonable definitions for these notions. Needless to say, the simplest dynamical behavior is periodicity. Because only long-time behavior really counts, we can agree that if \( x \) is asymptotically periodic, then its orbit is “regular” or “well behaved” from a dynamical perspective. Moreover, from the point of view of applications, small round-off errors are of no consequence, and since we are taking the “almost every point” approach, we can discard a small set of points from our study. Thus, adopting the terminology in [4, p. 136], we define a point \( x \) to be approximately periodic for \( f \) in \( C(I) \) if for each \( \epsilon > 0 \) there is a periodic point \( p \) such that

\[
\limsup_{n \to \infty} |f^n(x) - f^n(p)| < \epsilon.
\]

If \( \lambda \) signifies Lebesgue measure, then taking the following term to be synonymous with “almost everywhere” or “empirical” dynamical simplicity makes quite bit of sense:

Definition 2.1. A mapping \( f \) in \( C(I) \) is regular if \( \lambda \)-almost every point of \( I \) is approximately periodic.

It is worth emphasizing that this definition fits nicely into the older, purely topological framework. For instance, it can be proved that if \( x \) and \( y \) are approximately periodic

\[
\mu(A) = \lim_{n \to \infty} \frac{\text{card}\{0 \leq i \leq n - 1 : f^i(x) \in A\}}{n}.
\]

(Hence the larger—in the sense of \( \mu \)—the set \( A \) is the more often the orbits of almost all points visit it, and this rate does only depend on the set \( A \) itself.) The moral: while situation (a) is the phenomenon one would like to observe, even in case (b), where the dynamics become very wild, there are some probabilistic tools at hand that allow one to make far-reaching predictions.
for \( f \), then \( \{ x, y \} \) cannot be a scrambled set relative to this mapping [4, pp. 144–145]. In particular, if \( f \) is regular, then it cannot have any scrambled set of positive measure.

Moreover, if a map \( f \) has type less than \( 2^\infty \) in the Sharkovsky order, then all points are asymptotically periodic, whence \( f \) is regular. As these maps are regular in such a strong sense, one might be tempted to dismiss their dynamics (and the maps themselves) as trivial. Indeed, this is done so often that it is even difficult to find in available monographs—an exception is [4, pp. 121–122]—a detailed proof of the statement that we made at the beginning of this paragraph. In view of Sharkovsky’s theorem, it amounts to showing that if a map is of type 1, then all its orbits converge to fixed points. While elementary, the proof is rather instructive for a beginner in the subject (remarkably, although this fact has been known since at least 1953 [22], it has been reproved a number of times [7], [8], [37], [39], [43], even as recently as in [20]). It is based on the following result (see [4]):

**Proposition 2.2.** Let \( f \) in \( C(I) \) be of type 1, and let \( x \) be a point of \( I \). If \( f(x) > x \) (respectively, \( f(x) < x \)), then \( f^n(x) > x \) (respectively, \( f^n(x) < x \)) whenever \( n \geq 1 \).

In fact, consider a point \( x \) of \( I \). In order to establish the convergence of the sequence \( (f^n(x)) \) we can assume that the terms of this sequence are distinct, because if \( f^r(x) = f^{r+k}(x) \) for some numbers \( r \) and \( k \) with \( r \geq 0 \) and \( k \geq 1 \), then \( k = 1 \) and \( f^n(x) = f^r(x) \) whenever \( n \geq r \) (recall that \( f \) is of type 1). Also, we can suppose that \( (f^n(x)) \) is not eventually monotone, because in this case it converges to a point \( p \) that, in view of the continuity of \( f \), is a fixed point of \( f \).

Say, for instance, that \( f(x) > x \). Let \( n_1 \) be the first number \( n \) such that

\[
f^{n_1+1}(x) < f^n(x).
\]

Next, let \( n_2 \) be the first number \( n \) larger that \( n_1 \) such that

\[
f^{n_1+1}(x) > f^n(x),
\]

let \( n_3 \) be the first number \( n \) greater that \( n_2 \) such that

\[
f^{n_1+1}(x) < f^n(x),
\]

and so on. A repeated application of Proposition 2.2 to

\[
f^{n_1-1}(x), \quad f^{n_2-1}(x), \quad f^{n_3-1}(x), \ldots
\]

leads to

\[
x < f(x) < \cdots < f^{n_1-1}(x) < f^{n_2}(x) < f^{n_2+1}(x) < \cdots
\]

\[
< f^{n_3-1}(x) < f^{n_4}(x) < \cdots < \cdots < f^{n_4-1}(x) < \cdots
\]

\[
< f^{n_3+1}(x) < f^{n_3}(x) < f^{n_2-1}(x) < \cdots < f^{n_1+1}(x) < f^n(x). \quad (1)
\]

This implies that the orbit of \( x \) accumulates around two points \( p \) and \( q \), say with \( p \leq q \), satisfying \( f(p) = q \) and \( f(q) = p \). Since \( f \) has no periodic points of period two, we conclude that \( p \) and \( q \) are indeed the same fixed point of \( f \) and that the sequence \( (f^n(x)) \) converges to \( p \).

Enough of “order” for the moment. What about “chaos”? Recall that a definition of this notion is required to guarantee complex behavior for a “large” set of points.
A naive first attempt is to describe a map as “empirically chaotic” whenever it has a scrambled set of positive measure, but the “almost every point” specialist would raise his eyebrows at that definition, and not without reason: the full logistic map \( f(x) = 4x(1 - x) \) is known to behave quite wildly (for instance, almost all points have dense orbits in the interval), yet all its measurable scrambled sets have measure zero (see [18]).

The notion of sensitivity to initial conditions, introduced by Guckenheimer in his 1979 paper, is much more to the point. It is essentially the formal translation to the discrete setting of what is widely known as “the butterfly effect,” first introduced (in the setting of continuous-time models for climate prediction) by the meteorologist Lorenz in 1963. We quote from a Lorenz’s communication to the New York Academy of Sciences [24]: “One meteorologist remarked that if the theory were correct, one flap of a seagull’s wings would be enough to alter the course of the weather forever.” Some years later the seagull metamorphosed into a more poetic butterfly, and he titled his talk at the December 1972 meeting of the American Association for the Advancement of Science in Washington, D.C. “Predictability: Does the Flap of a Butterfly’s Wings in Brazil set off a Tornado in Texas?”

**Definition 2.3.** Let \( f \) belong to \( C(I) \) and let \( \delta > 0 \). We say that a point \( x \) of \( I \) is \( \delta \)-sensitive (with respect to the initial conditions) if for any neighborhood \( U \) of \( x \) there exists a positive \( n \) such that \( \text{diam}(f^n(U)) \geq \delta \). We call \( x \) sensitive if it is \( \delta \)-sensitive for some positive \( \delta \). Denote by \( S_f(\delta) \) (respectively, \( S_f \)) the set of \( \delta \)-sensitive (respectively, sensitive) points of \( f \). We pronounce \( f \) sensitive if \( \lambda(S_f(\delta)) > 0 \) for some \( \delta \) (or, equivalently, if \( \lambda(S_f) > 0 \)).

Now we are ready to state precisely the purpose of this paper, which is to investigate how the notions of regularity and sensitivity intertwine with each other. As the paper is addressed to a nonexpert audience, we provide “good old-fashioned” proofs requiring only elementary undergraduate-level prerequisites.

For instance, one might implicitly assume that a map cannot be “simple” and “complicated” at the same time, but . . . is this really so? Curiously enough, this very natural question has never been raised before, apparently having been taken for granted. We show, however, that there are maps in \( C(I) \) that are simultaneously regular and sensitive and that, surprisingly, such pathological examples can even arise in the most unexpected setting of all, namely, for certain maps of type 1 in the Sharkovskii ordering:

**Theorem 1.** There is a map \( f \) in \( C(I) \) of type 1 such that \( \lambda(S_f) = 1 \). In particular, there is a map \( f \) that is both regular and sensitive.

To be completely honest we are cheating a bit here. One of the key points of the “almost every approach” is that one should take into account only smooth maps, whereas our map \( f \) is not even differentiable. Indeed, in the setting of analytic maps things should work properly, for sensitive and regular maps are often characterized, respectively, by the property of having or not having an orbit whose set of limit points has nonempty interior. Unfortunately, the proof of this is far from elementary, but it is

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1Notice that the set \( S_f(\delta) \) is closed. Thus \( S_f \) is a Borel (hence Lebesgue measurable) set.

2It is a consequence of (among others) papers by Guckenheimer [14], Mahe [31], Lyubich [25], [26], Blokh and Lyubich [5], and Martens, de Melo, and van Strien [33], and holds true for the class of \( C^2 \)-maps with nonflat critical points (i.e., twice continuously differentiable maps \( f \) with the property that any point \( c \) for which \( f''(c) = 0 \) has a neighborhood \( V \) in which \( f \) is of class \( C^n+1 \) and \( f^{(n)}(c) \neq 0 \) for some \( n \geq 2 \) and without so-called wild attractors. This family includes, in particular, all logistic maps [27].
possible to show, at a very low technical cost, that for a large class of continuous maps of type less than $2^\infty$ paradoxical situations like that in Theorem 1 cannot arise at all. Specifically, the following is true:

**Theorem 2.** If $f$ in $N(I)$ is of type less than $2^\infty$, then $S_f$ is countable.

The class $N(I)$, which will be defined in section 4, includes all piecewise monotone mappings of $I$, as well as all continuously differentiable mappings of $I$ with the additional property of countably piecewise monotonicity. Here we say that $f$ in $C(I)$ is **piecewise monotone** if there is a partition

$$ P : 0 = a_0 < a_1 < \cdots < a_k = 1 $$

of $I$ such that $f_{[a_i, a_{i+1}]}$ is—not necessarily strictly—monotone for each $i$. By a **countably piecewise monotone** map we mean a map $f$ from $C(I)$ with the property that $f^{-1}(x)$ has at most countably many connected components for each $x$ in $I$. Observe that every piecewise monotone map is automatically countably piecewise monotone; on the other hand, a countably piecewise monotone map need not be piecewise monotone, even in the case when $f^{-1}(x)$ has finitely many components for each $x$ in $I$.

Until now we have used the words “large” and “small” in the measure-theoretic sense, but another approach, the topological one, is possible. From its vantage point, a set is large (respectively, small) if it is residual (respectively, of the first Baire category). Our third theorem shows that, from such a perspective, sensitivity is negligible for all continuous maps of type less than $2^\infty$.

**Theorem 3.** If $f$ in $C(I)$ is of type less than $2^\infty$, then $S_f(\delta)$ is nowhere dense for every positive $\delta$. In particular, the set $S_f$ is of the first Baire category.

Such contradictory situations as those underscored by Theorems 1 and 3 are not unusual in smooth one-dimensional dynamics. Moreover, they represent some of its most fascinating features. We recall the most famous example of this type of behavior: for the family of logistic maps the set of parameters $\alpha$ for which $f_\alpha$ is sensitive is of the first Baire category but has positive measure [13], [28], [17]. Notice that Theorem 3 also implies that Theorem 1 is sharp, in the sense that if $f$ is of type less than $2^\infty$, then no set $S_f(\delta)$ can have full measure (in fact, if $S_p(\delta)$ is dense in $I$, then it is the whole interval).

We establish Theorems 1, 2, and 3 in the next three sections. Throughout we denote by Int $A$ and Cl $A$ the interior and closure of a set $A$, respectively. We emphasize that topological notions always refer to the topology of $I$. Thus $[0, 1/2)$ is open and its boundary consists of the single point 1/2. If we write $A < B$ for sets $A$ and $B$, we mean that $x < y$ for each $x$ in $A$ and $y$ in $B$.

As (1) indicates, “point-by-point” dynamics can be described perfectly for maps of type 1. In view of Theorem 1, there is then a lesson to be learned: knowledge of the pointwise dynamics of a map does not necessarily carry with it a clear picture of the map’s global behavior. Striking as it might seem, there are almost no examples in the literature (except in the case of a unimodal map of type at most $2^\infty$; see, for instance, [45], [19]) where such a global description of the dynamics is attempted. Hence, for the sake of completeness, we have included section 6, which provides an effective structure theorem (see Theorem 6.9) for maps of type 1 in a new class $N^*(I)$ (still containing all piecewise monotone maps but also all maps of $I$ of class $C^1$) and is helpful in “visualizing” why Theorem 2 works. It must be stressed that if a map $f$ from $N(I)$ is of type 1, then $S_f$, while countable by Theorem 2, can still be dense in $I$.
On the other hand, although there are type-1 functions \( g \) in \( N^*(I) \) (even of class \( C^\infty \)) for which \( S_g \) has positive measure, an interesting consequence of the structure theorem is that the density of \( S_g \) is no longer possible:

**Theorem 4.** If \( f \) in \( N^*(I) \) is of type less than \( 2^\infty \), then \( S_f \) cannot be dense in \( I \).

While also elementary, the last section is rather longer and of a somewhat more involved nature than the previous ones. Since the proofs in the earlier sections are independent of it, the casual reader can skip it at the cost of missing only the proof of Theorem 4.

3. PROOF OF THEOREM 1. To prove the theorem we need only find a continuous function \( f : I \rightarrow I \) endowed with the following properties (by a *preimage* of a point \( x \) we mean any point \( y \) such that \( f^n(y) = x \) for some \( n \)):

(i) any point \( x \) of \( I \) has at most countably many preimages;
(ii) \( f(x) \geq x \) for each \( x \) in \( I \);
(iii) the set of preimages of 1 is dense in \( I \);
(iv) the set of preimages of the set \( C \) of fixed points of \( f \) has full measure.

Indeed, under these conditions we see that \( f \) is of type 1 (by (ii)). Because the set of preimages of \( C \setminus \{1\} \) has full measure (by (i) and (iv)) and is included in \( S_f \) (by (iii)), we are done.

Hence we want to construct a map \( f \) with the properties (i)-(iv). The set \( C \) of fixed points of \( f \) will be a Cantor set (meaning a closed, totally disconnected, perfect set) of positive measure. More precisely, we choose for \( m = 1, 2, \ldots \) partitions

\[
P_m : 0 = p_1^m < p_2^m < \cdots < p_{2^m+1}^m = 1
\]

of \( I \) satisfying

\[
p_{4i-3}^{m+1} = p_{2i-1}^m, \quad p_{4i}^{m+1} = p_{2i}^m
\]

for \( i = 1, 2, \ldots, 2^m \) in such a way that, writing

\[
I_i^m = [p_{2i-1}^m, p_{2i}^m]
\]

and defining

\[
C = \bigcap_{m \geq 1} \bigcup_{i=1}^{2^m} I_i^m,
\]

we obtain a set \( C \) of Cantor-type with an additional feature: there is some \( \alpha > 0 \) such that

\[
\frac{\lambda(I_i^m \cap C)}{\lambda(I_i^m)} > \alpha
\]  \hspace{1cm} (2)

for all \( i \) and \( m \). Now we define \( f \) on each of the intervals

\[
J_i^m = (p_{4i-2}^m, p_{4i-1}^m).
\]
Namely, \( f|_{J^m} \) consists of two affine pieces, the first one strictly increasing, the second one strictly decreasing, each reaching its maximum value \( r_i^m = f(q_i^m) \) at the midpoint \( q_i^m \) of \( J_i^m \). The numbers \( r_i^m \) are defined inductively, setting \( r_{2m-1} = 1 \) for each \( m \) and choosing \( r_i^m \) when \( i < 2^{m-1} \) so that the following statements hold:

(a) \( r_i^m \) belongs to the closest interval \( J_j^l \) with \( l < m \) that lies to the right of \( J_i^m \);
(b) \( r_i^m \) is mapped to \( q_j^l \) under some iterate of \( f \);
(c) if \( u \) and \( v \) are the right endpoint of \( J_i^m \) and the left endpoint of \( J_j^l \), respectively (that is, if \( [u, v] = [p_{2i-1}^m, p_{2j}^l] = I_{2j}^l \)), then \( r_i^m - v < v - u \)

(see Figure 1). We emphasize that, as a consequence of (b), all the points \( q_i^m \) are preimages of 1.

We must check that \( f \) is continuous and satisfies properties (i)–(iv). To begin with, (a) implies that

\[
f(J_i^m) \subset J_i^m \cup I_{2j}^l \cup J_j^l,\]

which together with (c) gives

\[
\lambda(f(J_i^m)) < \lambda(J_i^m) + 2\lambda(I_{2j}^l)\]
(this inequality is true even when \( i = 2m-1 \), since \( \lambda(f(J_i^m)) = \lambda(J_i^m) + \lambda(J_{m-i}^m) \) in this situation). Thus the lengths of the intervals \( f(J_i^m) \) tend to zero uniformly with respect to \( i \) as \( m \) tends to \( \infty \), which ensures the continuity of \( f \). Properties (i) and (ii) are immediate. Property (iii) follows once we have proved (iv), because for any closed interval \( K \) there is then a number \( l \) such that \( f^l(K) \cap C \neq \emptyset \) and, since \( f^l(K) \) is not degenerate (due to (i)), either \( f^l(K) \) itself or one of its iterates must contain one of the points \( q_i^m \), which is a preimage of 1. (More precisely, if

\[
f^l(K) \cap C \neq \emptyset
\]

and \( f^l(K) \) does not contain any of the points \( q_i^m \), then there is some interval

\[
J_i^{m'} = (p_{a_i'-2}^{m'}, p_{a_i'-1}^{m'})
\]

such that

\[
f^l(K) = [p_{a_i'-2}^{m'}, c]
\]

with \( c < q_{i'}^{m'} \) or

\[
f^l(K) = [d, p_{a_i'-1}^{m'}]
\]

with \( d > q_{i'}^{m'} \). In the first case there is an iterate of \( f^l(K) \) that contains \( q_{i'}^{m'} \); in the second case \( f^{l+1}(K) \) contains all points \( q_i^{m'} \) sufficiently close to \( p_{a_i'-1}^{m'} \).

Proving (iv) requires a preliminary construction. For any \( i \) and \( m \) let \( L_i^m \) and \( R_i^m \) be the components of \( J_i^m \setminus \{q_i^m\} \), labeled so that \( L_i^m < R_i^m \). First we claim that, if we fix \( i \) and \( m \) and if we write \( K = L_i^m \) or \( K = R_i^m \), then the following holds:

\[
K = \text{a disjoint union } D \cup (\cup K_r), \text{ where the set } D \text{ consists of preimages of } C, \lambda(K \cap D)/\lambda(K) > \alpha/2 \text{ with } \alpha \text{ as in (2), and the sets } K_r \text{ are open intervals (\*) homeomorphically and affinely mapped by iterates of } f \text{ onto some intervals } L_{i(r)}^m \text{ or } R_{i(r)}^m.
\]

Suppose first that \( K = R_i^m \). Then \( f(K) = (u, r_{i}^m) \) (we retain the same notation that occurs in (c)). Since \( [u, v] = L_u^m \), it is the union of a set \( E \) of preimages of \( C \) (recall that all the \( r_{i}^m \) are preimages of 1) and certain intervals \( L_{i}^m \) and \( R_{i}^m \). Furthermore, by (b), the interval \( (v, r_{i}^m) \) is mapped monotonically onto \( (v, q_{i}^m) = L_{i}^m \) after a finite number of iterations. From (2) we have \( \lambda([u, v] \cap E)/\lambda([u, v]) > \alpha \), while \( \lambda([v, r_{i}^m]) < \lambda([u, v]) \) by (c). Hence

\[
\frac{\lambda(f(K) \cap E)}{f(K)} = \frac{\lambda([u, v] \cap E)}{f(K)} > \frac{\lambda([u, v] \cap E)}{2\lambda([u, v])} > \frac{\alpha}{2}
\]

and (\*) follows because \( f \) is affine on \( K \). Notice that we are implicitly assuming here that \( i < 2^{m-1} \), since in the case \( i = 2^{m-1} \) (\*) follows directly from (2), even replacing \( \alpha/2 \) with \( \alpha \). To establish (\*) in the case \( K = L_i^m \) just take into account that we remove from the interval \( K \) a countable number of preimages of \( C \) to decompose it into a family of open intervals, each of which is mapped monotonically onto \( f(R_i^m) \) after a finite number of iterations.

We are now ready to finish the proof of (iv). Let \( P \) denote the set of preimages of \( C \). We must demonstrate that \( \lambda(P) = 1 \). We define inductively for \( z = 1, 2, \ldots \) a family of pairwise disjoint open intervals \( \{K_w^z\}_{w=1}^\infty \) with the property that for each \( w \) there exists \( l = l_w^z \) such that \( K_w^z \) is mapped affinely by \( f^l \) onto some interval \( L_i^m \) or \( R_i^m \). Namely, the family \( \{K_w^1\}_{w=1}^\infty \) consists of all intervals \( L_i^m \) and \( R_i^m \). To construct the fam-
ily \{K_{u}^{z+1}\}_{u=1}^{\infty}$ we take each of the intervals $K_{u}^{z}$ and replace it with the corresponding intervals $g^{-1}(K_{r})$, where $g = f^{z} \mid_{K_{r}}$ and $(\ast)$ is applied to $K = g(K_{r})$.

The construction clearly implies that

$$I \setminus P \subset \bigcup_{u=1}^{\infty} K_{u}^{z}$$

for any $z$. Moreover, by $(\ast)$ the sum of the lengths of the intervals $K_{u}^{z+1}$ that are included in a given interval $K_{u}^{z}$ is less than $(1 - \alpha/2)\lambda(K_{u}^{z})$. Thus

$$\sum_{u=1}^{\infty} \lambda(K_{u}^{z}) \leq (1 - \alpha/2)^{z-1}$$

for arbitrary $z$, from which it follows that $\lambda(P) = 1$.

4. PROOF OF THEOREM 2. When trying to understand why a map $f$ of type 1 can have a large set of sensitive points, one realizes immediately that the existence of repelling fixed points of $f$ having many (infinitely many!) preimages surely plays a prominent role. The examination of the map we constructed to prove Theorem 1 suggests that these preimages should accumulate at the corresponding fixed points but, as the map in Figure 2 dramatizes, a “long-distance effect” may also occur: for this function, the set of sensitive points consists of the preimages of the fixed point $p$ (a Cantor set of positive measure plus the point $p$ itself). Indeed, exploiting this idea, it is possible to construct a continuous function $g$ of type 1 satisfying $\lambda(S_{g}) = 1$ such that each of its fixed points $q$ has a neighborhood containing no preimage of $q$ other than $q$ itself (we will not go into the details). We emphasize that the map $f$ from Theorem 1 does not have this property. On the other hand, in contrast to $g$, it is countably piecewise monotone.

![Figure 2. For this map, all preimages of $p$ are sensitive.](image)

In this way we arrive quite naturally at the family of maps $N(I)$ under consideration in the theorem. We next give its precise definition.

Definition 4.1. A mapping $f$ in $C(I)$ is nice if the following conditions hold:
Remark 4.2. If a function \( f \) in \( C(I) \) is piecewise monotone or is continuously differentiable, then Definition 4.1(ii) holds. Let us explain why. In the first case, the piecewise monotonicity of \( f^n \) implies the existence of an interval \( J \) containing \( p \) and all points \( x_i \) near \( p \) such that \( f^n \) restricted to \( J \) is constant. In the second case, we use the fact that \( f^n \) is of class \( C^1 \) in tandem with Rolle’s theorem to conclude that \( (f^n)'(p) = 0 \). This implies that there is a small interval \( J \) containing \( p \) and all points \( x_i \) near \( p \) on which \( |(f^n)'| < 1 \), and with the additional property that \( f^n \) maps \( J \) into itself. Thus, in both cases, we find that the orbit of each point of \( J \) under the map \( f^n \) converges to \( p \).

In particular, all piecewise monotone maps and all countably piecewise monotone maps of class \( C^1 \) in \( I \) are nice. Thus, in a sense, any more or less “natural” mapping is nice, so Theorem 2 says that, unless a map \( f \) of type less than \( 2^\infty \) is rather “strange,” \( S_f \) is countable. We note in passing that the map in Figure 2 can be smoothed to make it differentiable or even of class \( C^\infty \) (which means that there exists a map \( h \) of type 1 and class \( C^\infty \) such that \( S_h \) has positive measure). Nevertheless, if a function \( f \) is continuously differentiable and has type less than \( 2^\infty \), then \( S_f \) cannot be dense in \( I \), hence cannot have full Lebesgue measure (see Theorem 4).

Finally, we observe that if \( f \) belongs to \( N(I) \), so does \( f^n \) for any \( n \). This is so because if \( f \) is countably piecewise monotone, then all its iterates are countably piecewise monotone as well (for the preimage of any open subset of \( I \) under any map in \( C(I) \) is open and therefore has at most countably many components).

Before proceeding with the proof, we offer a few preparatory words. If \( f \) is a member of \( C(I) \) and \( \delta > 0 \), then \( f(S_f(\delta)) \subseteq S_f(\delta) \). Hence \( f(S_f) \subseteq S_f \). Also, if \( f \) belongs to \( C(I) \), \( \delta > 0 \), and \( n \geq 1 \), then \( S_{f^n}(\delta) \subseteq S_f(\delta) \). If, additionally, \( \epsilon > 0 \) is sufficiently small, then \( S_f(\delta) \subseteq S_{f^n}(\epsilon) \) because of the uniform continuity of \( f \). Accordingly, \( S_f = S_{f^n} \) when \( n \geq 1 \). Thus, in order to prove Theorem 2, it is not overly restrictive to suppose that \( f \) is of type 1.

We must show that \( S_f \) is countable. Let \( P \) be the set of fixed points of \( f \). We claim that \( P \cap S_f \) is countable. Suppose not. Then there are a positive number \( \delta \) and an uncountable subset \( Q \) of \( P \cap S_f(\delta) \) with the property that, if \( p \) is an arbitrary point of \( Q \) and \( \epsilon \) is positive, then both \( Q \cap (p, p + \epsilon) \) and \( Q \cap (p - \epsilon, p) \) are uncountable. Next we use Definition 4.1(ii) to prove that \( f \) is strictly increasing at all points of \( Q \), that is, if \( p \) is in \( Q \) and \( x \) is sufficiently close to \( p \), then \( f(x) < p \) (respectively, \( f(x) > p \)) whenever \( x < p \) (respectively, \( x > p \)). In fact, taking a smaller (but still uncountable) set \( Q \) if necessary, we can even assume that there is a number \( \epsilon \) with \( \epsilon < \delta \) having the following property: if \( p \) belongs to \( Q \) and \( x \) is a point of the interval \( (p - \epsilon, p) \) (respectively, a point of the interval \( (p, p + \epsilon) \)), then \( f(x) < p \) (respectively, \( f(x) > p \)). Now realize that, if \( p \) and \( q \) are points of \( Q \) with \( 0 < q - p < \epsilon \), then \( f([p, q]) = [p, q] \). This is impossible because \([p, q]\) contains some (in fact, uncountably many) points from \( S_f(\delta) \).

Since \( f \) is countably piecewise monotone (recall Definition 4.1(i)), \( f^{-k}(P \cap S_f) \) has at most countably many components for each \( k \). If \( x \) belongs to the interior of one
of these components, then it cannot belong to $S_f$, for it has a neighborhood that is mapped to a point under some iterate of $f$. Now $f(S_f) \subset S_f$, so
\[ f^{-k}(P) \cap S_f \subset f^{-k}(P \cap S_f). \]

It follows that $S_f$ can contain at most countably many preimages of fixed points of $f$.

It remains to show that if $x$ in $I$ is such that all points $f^n(x)$ are different from the fixed point $p$ to which they converge, then $x$ does not lie in $S_f$. In this last part of the proof countably piecewise monotonicity is not needed.

Assume, to the contrary, that there is a $\delta$ such that $x$ lies in $S_f(\delta)$, recalling that all the points $f^n(x)$ and also $p$ belong to $S_f(\delta)$ (because $S_f(\delta)$ is closed). Suppose, for instance, that $f^n(x) < p$ for infinitely many $n$. Then there is a positive number $\epsilon$ such that $f(y) = p$ for no point $y$ in $(p - \epsilon, p)$. Indeed, Definition 4.1(ii) implies that, if $\epsilon_1$ is very small, the orbits of all points from $[p - \epsilon_1, p]$ converge to $p$. Thus, in order to comply with Proposition 2.2, if $y$ lies in $[p - \epsilon_1, p]$, then either $y \leq f^n(y) \leq p$ whenever $n \geq 1$ or $y < f^n(y) \leq f^m(y)$ for any such $n$, where $m$ is the first integer such that $f^m(y) > p$. As we can take $\epsilon_1$ small enough that
\[ [p - \epsilon_1, p] \cup f([p - \epsilon_1, p]) \subset (p - \delta/2, p + \delta/2), \]

we get
\[ f^n([p - \epsilon_1, p]) \subset (p - \delta/2, p + \delta/2) \]

for any $n$, in contradiction with the fact that $(p - \epsilon_1, p) \cap S_f(\delta) \neq \emptyset$.

Furthermore, if $\epsilon < \delta$, then $(p - \epsilon, p)$ contains no fixed points of $f$. If $q$ is a fixed point belonging to this interval and $f^m(x)$ lies between $q$ and $p$, then $f^n((f^m(x), p)) \subset (q, p)$ for each $n$, as we prove shortly, contradicting the fact that the interval $(f^m(x), p)$ intersects $S_f(\delta)$.

We must justify the assertion that $f^n((f^m(x), p))$ is a subset of $(q, p)$ for each $n$. We proceed inductively. Observe that the data $f(q) = q$ and $f(y) \neq p$ for all points $y$ in $(p - \epsilon, p)$ imply that $f(y) < p$ for each such $y$. Thus, if our statement fails for $n = 1$, there is a point $z$ lying between $f^m(x)$ and $p$ such that $f(z) = q$ (hence $f^k(z) = q$ for each $k$). In particular, if $k$ is sufficiently large, the relation $f^k((f^m(x)) > z$ guarantees the existence of points $q_0, q_1, z_1$, and $z_0$, with
\[ q \leq q_0 < q_1 < z_1 < z_0 \leq z, \]
satisfying
\[ f^k([q_0, q_1]) = f^k([z_1, z_0]) = [q, z]. \]

We can then find a subinterval $J$ of $[q_0, q_1]$ for which $f^k(J) = [z_1, z_0]$. Notice that $f^{2k}(J)$ contains $J$, so there is a point $w$ in $J$ with the property $f^{2k}(w) = w$. This point must be a fixed point of $f$ because $f$ is of type 1. Hence $f^k(w) = w$, which contradicts the disjointness of $J$ and $f^k(J)$.

Assume now that $f^n((f^m(x), p)) \subset (q, p)$ for some $n$. Then $f^{n+1}(y) < p$ for any $y$ belonging to $(f^m(x), p))$. If $f^{n+1}((f^m(x), p))$ is not a subset of $(q, p)$, then we find a point $z'$ between $f^m(x)$ and $p$ such that $f^k(z') = q$ for every $k$ with $k \geq n + 1$, and we arrive at a contradiction in a fashion similar to the way that we did in the first step of the induction.
We have shown that \( f(y) \neq p \) and \( f(y) \neq y \) for each point \( y \) of \((p - \epsilon, p)\). Hence three possibilities arise. The first one, that \( f(y) < y \) for each \( y \) in \((p - \epsilon, p)\), can be rejected immediately. It implies that, if \( f^n(x) \) belongs to \((p - \epsilon, p)\), then \( f^m(x) < p - \epsilon \) for some \( m \) with \( m > n \), which is impossible because the sequence \( (f^n(x)) \) converges to \( p \) and infinitely many of its points lie in \((p - \epsilon, p)\). The second one, that \( y < f(y) < p \) for each \( y \) in \((p - \epsilon, p)\), leads to the contradiction

\[
(p - \epsilon, p) \cap S_f = \emptyset.
\]

Thus we must have \( f(y) > p \) for every \( y \) in \((p - \epsilon, p)\), which in particular implies that there are infinitely many points from the sequence \( (f^n(x)) \) to the right of \( p \). By invoking a right-handed version of our previous argument, we realize that it entails no restriction to assume that \( f(y) < p \) for \( y \) in \((p, p + \epsilon)\) as well.

We have found an \( \epsilon > 0 \) such that \( f(y) > p \) for all \( y \) in \((p - \epsilon, p)\) and \( f(y) < p \) for every \( y \) in \((p, p + \epsilon)\). Appealing again to Proposition 2.2, we conclude that if \( y \) is close enough to \( p \), then its whole orbit lies in the interval with endpoints \( y \) and \( f(y) \), hence is also very close to \( p \). In particular, \( p \) cannot belong to \( S_f(\delta) \), so we have arrived at a contradiction. Theorem 2 is now established.

5. PROOF OF THEOREM 3. To prove Theorem 3 we require two lemmas. They imply that maps \( f \) of type 1 have a kind of “shrinking” character. Both lemmas are concerned with invariant intervals of such maps. (By an invariant interval of \( f \) we mean an interval \( J \) such that \( f(J) \subseteq J \).)

**Lemma 5.1.** If \( f \) in \( C(I) \) is of type 1 and \( J \) is a closed invariant interval of \( f \), then there is a closed invariant interval \( T \) that is properly contained in \( J \).

**Proof.** As \( f | J \) has no dense orbits, there is a subinterval \( K \) of \( J \) such that

\[
\text{Cl} \left( \bigcup_{n=0}^{\infty} f^n(K) \right) \neq J
\]

(this is an easy exercise; alternatively, see [4, p. 155]). For this interval \( K \), two possibilities arise: either the intervals \( f^n(K) \) \((n = 0, 1, 2, \ldots)\) are pairwise disjoint or there are integers \( k \) and \( l \) with \( k < l \) such that \( f^k(K) \cap f^l(K) \neq \emptyset \). The second case is the easier one: in it we get \( f^{l-k}(R) \subseteq R \) for the interval

\[
R = \text{Cl} \left( \bigcup_{i=0}^{\infty} f^{k+i(l-k)}(K) \right).
\]

Then there is some \( p \) in \( R \) such that \( f^{l-k}(p) = p \). Because \( f \) is of type 1, \( p \) is a fixed point of \( f \), \( f^n(R) \cap f^{n+1}(R) \neq \emptyset \) for all \( n \), and

\[
T = \text{Cl} \left( \bigcup_{n=0}^{\infty} f^n(R) \right) = \text{Cl} \left( \bigcup_{n=k}^{\infty} f^n(K) \right)
\]

is an interval of the type we seek.

Now assume that the intervals \( f^n(K) \) are pairwise disjoint, and let \( K' (K \subseteq K' \subseteq J) \) be the maximal interval with the property that the intervals \( f^n(K') \) are pairwise disjoint. We can even assume that the intervals \( \text{Cl} f^n(K') \) are pairwise disjoint, for if

\[
\text{Cl} f^k(K') \cap \text{Cl} f^l(K') \neq \emptyset
\]
for some $k$ and $l$ with $k < l$, then we can take

$$R' = \text{Cl} \left( \bigcup_{i=1}^{\infty} f^{k+i(l-k)}(K') \right), \quad T = \text{Cl} \left( \bigcup_{n=0}^{\infty} f^n(R') \right)$$

again (notice that $\text{Int}(f^k(K')) \cap T = \emptyset$). We now appeal to Proposition 2.2 to find integers $m, u$, and $t$ ($m \geq 0$, $1 \leq u < t$) such that either

$$f^m(K') < f^{m+u}(K') < f^{m+t}(K')$$

or

$$f^{m+t}(K') < f^{m+u}(K') < f^m(K').$$

Say, for instance, that the first possibility holds. As the intervals $\text{Cl} f^n(K')$ are pairwise disjoint, we can find a subinterval $K''$ of $J$ properly containing $K'$ so that

$$f^m(K'') < f^{m+u}(K'') < f^{m+t}(K'').$$

Again by Proposition 2.2, all the intervals $f^n(K'')$ ($n \geq m + t$) lie to the right of $f^m(K'')$. Since, on the other hand, the maximality of $K'$ implies the existence of $k$ and $l$ with $k < l$ such that $f^k(K'') \cap f^l(K'') \neq \emptyset$ and we may assume that $m + t \leq k$, the sets

$$R'' = \text{Cl} \left( \bigcup_{i=0}^{\infty} f^{k+i(l-k)}(K'') \right), \quad T = \text{Cl} \left( \bigcup_{n=0}^{\infty} f^n(R'') \right)$$

do the job (because, for instance, $f^m(K'') \cap T = \emptyset$).

The second lemma ensures the existence of invariant intervals at arbitrarily small scales.

**Lemma 5.2.** If $f$ in $C(I)$ is of type 1 and $\delta > 0$, then $f$ has an invariant subinterval of length less than $\delta$.

**Proof.** Assume to the contrary that there is a positive number $\delta$ such that all elements of the set $\mathcal{I}$ of closed invariant intervals of $f$ have length at least $\delta$. If we order this set by inclusion, then we see that any decreasing chain in $\mathcal{I}$ has a lower bound, so we can apply Zorn’s lemma to find a minimal closed invariant interval $J$. This contradicts Lemma 5.1.

In the proof of Theorem 3 we will make use of the principle of transfinite induction. Just in case the reader is not familiar with it we formulate a simplified version that suffices for our purposes. We refer readers to [32, pp. 95–99] for the details.

The family $\mathcal{O}$ of countable ordinals can be described informally as follows. The initial elements of $\mathcal{O}$ are the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, ..., but the reader will surely prefer the more familiar names $0, 1, 2, 3, \ldots$ for them. The first infinite ordinal is $\aleph_0 = \{0, 1, 2, 3, \ldots\}$, next follow $\aleph_0 + 1 = \{0, 1, 2, 3, \ldots, \aleph_0\}$, then $\aleph_0 + 2 = \{0, 1, 2, 3, \ldots, \aleph_0, \aleph_0 + 1\}$, etc. It turns out that inclusion defines a total ordering in $\mathcal{O}$ (thus, if $\alpha$ and $\beta$ belong to $\mathcal{O}$ and we say that $\alpha$ is smaller than $\beta$, then
we mean that \( \alpha \) is included in \( \beta \). We remark that, although \( \mathcal{O} \) consists of countable sets, \( \mathcal{O} \) itself is uncountable.

It can be proved that if a subset \( A \) of \( \mathcal{O} \) contains 0 and has the property that whenever it contains all the ordinals smaller than a given ordinal \( \alpha \) it also contains \( \alpha \), then \( A \) is the whole set \( \mathcal{O} \). This means that we can establish properties \( \Phi(\alpha) \) for all ordinals \( \alpha \) in \( \mathcal{O} \) and define families of sets \( \{J_\alpha\}_{\alpha \in \mathcal{O}} \) in exactly the same spirit as when we use the standard principle of finite induction. In particular, to define a certain family of sets \( \{J_\alpha\}_{\alpha \in \mathcal{O}} \), it is sufficient to define \( J_0 \) and then explain how the set \( J_\alpha \) must be defined in terms of the previously defined sets \( J_\beta \) with \( \beta < \alpha \).

**Proof of Theorem 3.** It again entails no loss of generality to assume that \( f \) is of type 1. Let \( \delta \) be given. We exploit transfinite induction to define an open interval \( J_\alpha \) for each countable ordinal \( \alpha \) such that the following statements hold:

(i) \( S_f(\delta) \subset I \setminus \bigcup_{n=0}^{\infty} f^{-n}(J_\alpha) \);

(ii) if \( C_\alpha = I \setminus \bigcup_{\beta < \alpha} \bigcup_{n=0}^{\infty} f^{-n}(J_\beta) \) has nonempty interior, then the intervals \( J_\beta \) with \( \beta < \alpha \) are pairwise disjoint.

From this, the theorem follows easily, for at least one of the sets \( C_\alpha \) must have empty interior (otherwise we would arrive at the contradiction that there is an uncountable family \( \{J_\alpha\} \) of pairwise disjoint intervals).

For the interval \( J_0 \) we choose an arbitrary interval of length less than \( \delta \) with invariant closure (see Lemma 5.2). Now let \( \sigma \) be a positive countable ordinal, and assume that we have defined the intervals \( J_\alpha \) for all \( \alpha \) with \( \alpha < \sigma \) so that (i) and (ii) hold. We show how to extend this to the case \( \alpha = \sigma \). If \( \operatorname{Int} C_\alpha = \emptyset \), then we simply take \( J_\alpha = J_0 \). If \( \operatorname{Int} C_\alpha \neq \emptyset \), then we take an open subinterval \( J \) of \( \operatorname{Int} C_\alpha \), giving rise to two possibilities. If all the intervals \( f^n(J) \) are pairwise disjoint, then their lengths tend to zero and we can put \( J_\sigma = J \). If not, then we argue as in the first paragraph of the proof of Lemma 5.1 to find a closed invariant interval \( T \) of \( f \) with \( T \subset \bigcup_{n=0}^{\infty} f^n(J) \subset C_\alpha \) that, in light of Lemma 5.2, we can assume to have length less that \( \delta \). After removing the endpoints from \( T \), we obtain the required interval \( J_\sigma \).

**6. A STRUCTURE THEOREM FOR MAPS OF TYPE 1.** In this section we show that the dynamical structure of a map \( f \) in \( N^*(I) \) of type 1 can be described in a very effective way. From this information an alternative and more “visual” proof of Theorem 2 (for countably piecewise monotone maps in \( N^*(I) \)) will emerge. The motivated reader should find no difficulty in producing the analogues of our results in the most general case, the case where \( f \) has type less than \( 2^\infty \). The family \( N^*(I) \), a modification of the family \( N(I) \) of nice maps, consists of all mappings in \( C(I) \) satisfying (ii) in Definition 4.1 but

(i\(^*\)) if \( n \) is a positive integer and \( (p_i)_{i=1}^{\infty} \) is a convergent sequence of fixed points of \( f^n \), then there is some \( j \) such that \( f^n([p_i; p_{i+1}]) = [p_i; p_{i+1}] \) whenever \( i \geq j \) (here \( [a; b] \) denotes the smallest connected set containing \( \{a, b\} \))

instead of (i). It is easy to check that (i\(^*\)) is satisfied by all piecewise monotone maps and all continuously differentiable maps of \( I \). Hence \( N^*(I) \), in contrast to \( N(I) \), contains all functions of class \( C^1 \).

We recall for the reader the notion of “immediate basin of attraction.” We seize the opportunity to assemble other concepts and facts that will later prove their value.

**Definition 6.1.** Let \( f \) belong to \( C(I) \), and let \( p \) be a fixed point of \( f \). We say that \( p \) is an attractor (with respect to \( f \)) if the set of points whose orbits tend to \( p \) in-
cludes a nondegenerate (maximal) interval containing $p$. This interval is called the *immediate basin of attraction* of $p$ and is denoted by $B_f(p)$. If $f(B_f(p)) = B_f(p)$ and the sequence $(f^n(x))$ is monotone for each $x$ in $B_f(p)$, then we call $p$ a trivial attractor; otherwise it is nontrivial. If $p$ is an attractor different from either of the endpoints of $B_f(p)$, then we refer to $p$ as a bilateral attractor. Similarly, if $J$ is a compact, $f$-invariant subinterval of $I$, then we define the *immediate basin of attraction* $B_f(J)$ of $J$ to be the maximal interval containing $J$ with the property that the orbits of all its points eventually fall into $J$.

**Remark 6.2.** Notice that if $p$ is a trivial attractor, then one of the following alternatives must occur:

(a) $B_f(p) = [p, r)$ for some fixed point $r$ of $f$, and $p \leq f(x) < x$ holds for each $x$ in $(p, r)$;

(b) $B_f(p) = (q, p]$ for some fixed point $q$ of $f$, and $x < f(x) \leq p$ holds for each $x$ in $(q, p)$;

(c) $B_f(p) = (q, r)$ for fixed points $q$ and $r$ of $f$, and $p \leq f(x) < x$ (respectively, $x < f(x) \leq p$) holds for each $x$ in $(p, r)$ (respectively, for each $x$ in $(q, p)$).

We emphasize that the map $f$ itself need not be monotone on $B_f(p)$.

On the other hand, if $p$ is a nontrivial attractor and $f(B_f(p)) = B_f(p)$, then it is a simple matter to check that there exists a positive $\epsilon$ such that $f(x) > x$ for every $x$ in $(p - \epsilon, p)$ and $f(x) < x$ for every $x$ in $(p, p + \epsilon)$. Figures 3 and 4 provide examples of trivial and nontrivial attractors, respectively, for the indicated functions. Observe

![Figure 3](image3.png)  
**Figure 3.** In all three cases $p$ is a trivial attractor.

![Figure 4](image4.png)  
**Figure 4.** In cases $p$ is a nontrivial attractor.
that the only reason why the point $p$ in the left-hand picture in Figure 4 fails to be a trivial attractor is that $f(B_f(p))$ is strictly contained in $B_f(p)$.

**Definition 6.3.** Let $f$ be a member of $C(I)$, and let $J = [a, b]$ be a subinterval of $I$. We speak of $J$ as neutral (with respect to $f$) if the following conditions obtain:

(i) $f([x, y]) = [x, y]$ for any pair of consecutive fixed points $x$ and $y$ ($x < y$) of $f$ in $J$ (to say that $x$ and $y$ are consecutive fixed points of $f$ means that $(x, y)$ does not contain a fixed point);

(ii) $a$ and $b$ are fixed points of $f$;

(iii) neither $a$ nor $b$ is a bilateral attractor for $f$.

In Figure 5 the interval $[a, b]$ from the left-hand picture is a neutral interval for the function depicted. However, in the right-hand picture this is not the case because $a$ is a bilateral attractor (but $[a', b]$ is neutral).

![Figure 5. Some neutral and nonneutral intervals.](image)

We formulate our structure theorem in terms of “atoms” and “molecules” of different “levels.” Atoms of level 0 are, so to speak, the smallest pieces contributing to the dynamics: immediate basins of nontrivial attractors and maximal neutral intervals. We construct our molecules by gluing together atoms of the same level that are not pairwise disjoint, while atoms of level $n$ are nothing but the immediate basins of attraction of molecules of level $n - 1$. As it turns out (this is the conclusion of Theorem 6.9), if $f$ is a type 1-map from $N^*(I)$, then the whole interval $I$ is an atom of some level $l$.

**Definition 6.4.** Let $f$ belong to $C(I)$, and let $J$ be a compact subinterval of $I$.

(i) $J$ is an *atom of level 0* of $f$ if $J$ is either a maximal neutral interval or the closure of the immediate basin of attraction of a nontrivial attractor.

(ii) $J$ is a *molecule of level 0* of $f$ if it is the union of a finite number of atoms of level 0 and if it is maximal with respect to this property.

(iii) $J$ is an *atom of level* $n(\geq 1)$ of $f$ if it is the closure of the immediate basin of attraction of a molecule of level $n - 1$. 

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(iv) \( J \) is a molecule of level \( n \geq 1 \) of \( f \) if it is the union of a finite number of atoms of level \( n \) and if it is maximal with respect to this property.

Notice that the definition makes sense because all atoms and molecules are clearly invariant under \( f \) and that the boundary of any atom of \( f \) is mapped into itself by \( f \).

Figure 6 illustrates the structure in terms of atoms and molecules of a map of type 1 belonging to \( N^\times(I) \). The map \( f \) here has three atoms of level 0: \( A_i^0 \) is a maximal neutral interval; \( A_i^0 \) and \( A_i^0 \) are the closures of the immediate basins of attraction of the fifth and eighth fixed points of the map, respectively. As \( A_i^0 \) and \( A_i^0 \) have a common point they combine to form one molecule \( M_i^0 \) of level 0, while \( M_i^2 = A_i^0 \). The atoms of level 1 are the closures of the basins of attraction of these molecules—\( A_i^1 \) and \( A_i^2 \), respectively—which together form \( M_i^1 \), the only molecule of level 1. It comprises the whole interval \([0, 1]\). Of course, its basin of attraction \( A_i^2 \) is also the whole interval itself, hence is an atom of level 2.

![Figure 6. Atoms and molecules for a map of type 1.](image)

Now the need for modifying the definition of our initial family \( N(I) \) becomes apparent: the map \( f \) in Figure 7 is nice but has no atoms at all. By the way, if the local maxima of \( f \) are carefully chosen, then its set \( S_f \) of sensitive points can be made dense in \( I \) (compare with Theorem 4). We will not go into the cumbersome details.
Before proceeding to the proof of Theorem 6.9 we assemble some detailed information on attractors and neutral intervals. Our first lemma can be seen as a kind of converse to the last statement in Remark 6.2.

**Lemma 6.5.** Let $f$ in $C(I)$ be of type 1, and let $p$ be a fixed point of $f$. If there is some positive $\epsilon$ such that $f(x) > x$ for each $x$ in $(p - \epsilon, p)$ and $f(x) < x$ for each $x \in (p, p + \epsilon)$, then $p$ is a bilateral attractor.

*Proof.* Let $\epsilon_1 (\leq \epsilon)$ be such that $|f(x) - p| < \epsilon$ whenever $|x - p| < \epsilon_1$. We demonstrate that if $|x - p| < \epsilon_1$, then $(f^n(x))$ tends to $p$.

Let $q$ be the limit fixed point of the sequence $(f^n(x))$ and assume that $p - \epsilon_1 < x < p$ (the other case is similar). If $f^n(x) \leq p$ for each $n$, then $(f^n(x))$ is increasing and $p = q$ follows ($p$ is the only fixed point in the interval $(p - \epsilon, p + \epsilon)$). If $k$ is the first integer such that $f^k(x) > p$, then we infer from the definition of $\epsilon_1$ that $f^k(x) < \epsilon$. Now

$$f^n(x) \in [f^{k-1}(x), f^k(x)] \subset (p - \epsilon, p + \epsilon)$$

whenever $n \geq k$ (Proposition 2.2), so again $p = q$. 

**Lemma 6.6.** The following statements hold for each $f$ in $C(I)$:

(i) any neutral interval for $f$ is contained in a maximal one;

(ii) if $p$ is a trivial attractor of $f$, then $\text{Cl}(B_f(p))$ is a neutral interval;

(iii) if $J$ is a neutral interval for $f$ and if a point $p$ of $J$ is an attractor for $f$ that is not an endpoint of $J$, then it is trivial and $B_f(p)$ is contained in $J$;

(iv) if $f$ belongs to $N^*(I)$ and $P$ is an infinite set of fixed points of $f$, then there is a neutral interval of $f$ that contains infinitely many points from $P$.

*Proof.* Statement (i) follows from the fact that, if $J$ is neutral for $f$ and $K$ is the union of all neutral intervals containing $J$, then $\text{Cl}(K)$ is clearly neutral as well (thus $K = \text{Cl} K$ is one such interval). Statement (ii) follows from Remark 6.2.
To prove (iii), assume that there are no nearby fixed points to the right of $p$ but infinitely many to its left, these accumulating at $p$ (the other possibilities can be dealt with in similar fashion). Then we exploit the fact that $J$ is neutral for $f$ to find a fixed point $r$ with $r > p$ such that $p$ and $r$ are consecutive fixed points. We then get $B_f(r) = \{p, r\}$ and arrive at the desired conclusion.

To establish (iv), assume, for example, that $p$ is an accumulation point of a subset $P'$ of $P$ that lies to the left of $p$ and fix $q$ in $P'$ close enough to $p$ so that $f([s, r]) = [s, r]$ for any pair of fixed points $s$ and $r$ ($s < r$) in $[q, p]$ (this is possible by (i*) in the definition of $N^*(I)$). As neither $p$ nor $q$ is a bilateral attractor, then $[q, p]$ does the job. If $q$ is a bilateral attractor and $q'$ is the consecutive fixed point to the right of $q$, then $q \leq f(x) < x$ for each $x$ in $(q, q')$. Thus $q'$ is not a bilateral attractor and $[q', p]$ is the neutral interval we desire. ■

The next two lemmas clarify the situation as to the number of atoms that a map can have and their relative positions.

**Lemma 6.7.** If $f$ belongs to $C(I)$, then two different atoms of $f$ of the same level have at most one common (fixed) point.

**Proof.** Let $J_1$ and $J_2$ be different atoms of level $n$. We suppose $\text{Int}(J_1 \cap J_2) \neq \emptyset$ and argue to a contradiction.

Consider first the case $n = 0$. It is clearly impossible that both $J_1$ and $J_2$ are the closures of immediate basins of attraction of respective fixed points $p_1$ and $p_2$, for all points of $\text{Int}(J_1 \cap J_2)$ would then be attracted by both $p_1$ and $p_2$. Similarly, if both $J_1$ and $J_2$ are neutral intervals for $f$, then we find that $J_1 \cup J_2$ is neutral as well, which leads to $J_1 = J_1 \cup J_2 = J_2$ and thus contradicts the maximality of $J_1$ and $J_2$.

Assume that $J_1$ is neutral, and let $p$ be the fixed point of $f$ attracting the orbits of all points in $\text{Int} J_2$. Since $f(J_1 \cap J_2) \subset J_1 \cap J_2 \subset J_1$ and all orbits of points in $\text{Int}(J_1 \cap J_2)$ are attracted by $p$, we must have $p$ in $J_1$. According to Lemma 6.6(iii) $p$ must then be one of the endpoints of $J_1$, say the left one. Let $r$ be the nearest fixed point of $f$ to the right of $p$. Such a point exists because all points from $\text{Int}(J_1 \cap J_2)$ are attracted by $p$, hence cannot be fixed by $f$. We infer that $f([p, r]) = [p, r]$. Putting together the assumption that $\text{Int}(J_1 \cap J_2) \neq \emptyset$ with the fact that $p$ is not a bilateral attractor (because $J_1$ is neutral), we conclude that $B_f(p) = \{p, r\}$ and that $p$ is a trivial attractor, a contradiction. This disposes of the case $n = 0$.

Suppose now that $n \geq 1$ and that $J_1$ and $J_2$ are the closures of the immediate basins of attraction of different molecules of level $n - 1$, say $K_1$ and $K_2$, respectively. From the definition it is obvious that $K_1 \cap K_2 = \emptyset$. This is impossible, however, because if $x$ belongs to $\text{Int}(J_1 \cap J_2)$ and $m$ is suitably large, then $f^m(x)$ belongs to both $K_1$ and $K_2$. ■

**Lemma 6.8.** If $f \in N^*(I)$ is of type $1$, then $f$ has a positive, finite number of atoms of level $0$.

**Proof.** For starters we show that $f$ has at least some atom of level $0$. In view of Lemma 6.6(i) and (iv) we can assume that $f$ has a finite number of fixed points, which we list as $0 \leq p_1 < p_2 < \cdots < p_k \leq 1$. Since any attractor is included in some atom of level $0$ (this follows from the definition if the attractor is nontrivial and from Lemma 6.6(i) and (ii) if it is trivial), it suffices to show that at least one of the points $p_i$ is an attractor. This is trivial in the case $k = 1$: all orbits are attracted to the unique fixed point of $f$. Assume that $k > 1$. Notice that if $f(x) < x$ for every $x$ in $(p_1, p_2)$,
then \( \{J_n\}_{n=1}^\infty \) of atoms of level 0, which we can suppose to be pairwise disjoint (Lemma 6.7). Select a fixed point \( p_0 \) of \( f \) in \( J_n \) for each \( n \), and consider the set \( P = \{p_n\} \). Referring to Lemma 6.6(iv) and Lemma 6.7 we arrive at a contradiction.

Finally, everything is in place to prove:

**Theorem 6.9.** If \( f \in N^*(I) \) is of type 1, then \( I \) is an atom of \( f \) of some level \( l \).

**Proof.** According to Lemma 6.6(i) and (iv) (see also Lemma 6.8), the set \( P_m \) of fixed points of \( f \) not belonging to the interior of some atom of level \( m \) is finite. It suffices to show that \( \text{card} \ P_m > \text{card} \ P_{m+1} \) unless \( P_m = \emptyset \), for then \( P_l = \emptyset \) must hold for some \( l \).

Since by Lemma 6.8 \( f \) must have an atom of level \( l \), the only possibility is that \( f \) has exactly one atom of level \( l \), the interval \( I \) itself (recall that \( f \) maps the boundary of every atom to itself).

Suppose then that \( \text{card} \ P_m > 0 \). If two atoms of level \( m \) share a common point (see Lemma 6.7), then it is a fixed point of \( f \) and trivially this point cannot belong to \( P_{m+1} \). Hence we may as well assume that all atoms of level \( m \) are pairwise disjoint (i.e., each atom of level \( m \) is also a molecule of level \( m \)). Let \( J_i = [p_i, q_i] \) (\( i = 1, 2, \ldots, r \)) be the atoms of level \( m \) of \( f \), labeled so that \( J_i < J_{i+1} \) for each \( i \). We show that one of the points \( p_i \) or \( q_i \) is fixed and belongs to the interior of the atom of level \( m + 1 \) containing \( J_i \).

Assume initially that \( 0 < p_1 \) and that \( p_1 \) is fixed by \( f \). Then \( f(x) \geq x \) must hold for all \( x \) in \([0, p_1]\); if not, we could reason as in the proof of Lemma 6.8 to conclude that one of the (finitely many) fixed points to the left of \( p_1 \) is an attractor, which is impossible. We claim that if \( \epsilon > 0 \) is small enough, then \( p_1 < f(x) < q_1 \) for any \( x \) in \((p_1 - \epsilon, p_1)\). Otherwise either \( f(x) < p_1 \) for all \( x \) sufficiently close to \( p_1 \) with \( x < p_1 \), or there is a sequence \( (x_n) \) converging to \( p_1 \) such that \( x_n < p_1 \) and \( f(x_n) = p_1 \) for every \( n \). In each case \( p_1 \) is an attractor and \( B_f(p_1) \) contains a small interval to the left of \( p_1 \), which we have ruled out (notice that we are using Definition 4.1(ii) for the first time in the proof). It follows that \((p_1 - \epsilon, q_1)\) is included in the atom of level \( m + 1 \) containing \( J_1 \), so we are done.

Now consider the situation where either \( p_1 = 0 \) or \( p_1 \) is not fixed by \( f \). Since \( J_1 \) is not the whole interval \( I \) and since \( f \) preserves the boundaries of atoms, \( q_1 < 1 \) and \( q_1 \) is fixed. Moreover, we can assume that \( r \geq 2 \), for otherwise an argument similar to the previous one finishes the proof. Next, observe that if \( p_2 \) is a fixed point of \( f \), then we must have \( f(x) \geq x \) for every \( x \) in \([q_1, p_2]\) in order to preclude the existence of attractors in the interval \([q_1, p_2]\) (and exclude the possibility that, in the case where \( q_1 \) is an attractor, \( B_f(q_1) \) contains some points to the right of \( q_1 \)). Now we reason as in the previous paragraph to find an atom of level \( m + 1 \) containing \( p_2 \) in its interior.

Thus we can assume that \( q_2 < 1 \) and that \( q_2 \) is fixed by \( f \). Repeating the foregoing argument, we reach the desired conclusion after a finite number of steps.

To close the paper we prove Theorem 4 and show how Theorem 2 can be derived easily (for countably piecewise monotone maps in \( N^*(I) \)) from Theorem 6.9. We need a further lemma:
Lemma 6.10. If \( f \) belongs to \( C(I) \) and \( J \) is an atom of \( f \) of level 0, then \( J \cap S_f \) is countable and \( S_f \) cannot be dense in \( J \).

Proof. Let \( J \) be an atom of level 0 of \( f \). If \( J \) is the closure of the immediate basin of attraction of some attractor, then none of its interior points is sensitive. This follows readily from Remark 6.2 if the attractor is trivial, and it is implicit in the proof of Lemma 6.5 if the attractor is nontrivial. If \( J \) is neutral, then by Definition 6.3(i) all sensitive points of \( J \) must be fixed by \( f \). Moreover, if a fixed point of \( f \) in \( J \) is sensitive and it is not an endpoint of \( J \), then again because of Definition 6.3(i) it must be the endpoint of a subinterval of \( J \) not containing any other fixed point of \( f \). Therefore \( J \cap S_f \) is countable as well. Observe finally that \( S_f \) cannot be dense in \( J \) in either case.

We want to verify that if \( f \) in \( N^*(I) \) is of type less than 2\(^\infty \), then \( S_f \) cannot be dense (Theorem 4) and that, under the additional hypothesis of countably piecewise monotonicity for \( f \), it is countable (Theorem 2). As earlier, it suffices to treat the case where \( f \) is of type 1. In fact, Theorem 4 follows immediately from Theorem 6.9 (or Lemma 6.8) and Lemma 6.10.

To prove Theorem 2 we establish that any atom of \( f \) has at most countably many sensitive points. After this we can appeal to Theorem 6.9 to finish the job.

We proceed by induction on the level of atoms of \( f \). The statement follows from Lemma 6.10 for atoms of level 0. Now assume that the set of sensitive points contained in any atom (and then in any molecule) of level \( m \) is countable. Let \( J \) be an arbitrary atom of level \( m + 1 \), and let \( M \) be the molecule of level \( m \) such that \( J = \text{Cl} B_f(M) \). If \( x \) belongs to \( S_f \cap B_f(M) \), then there is some \( k = k_x \) such that \( f^k(x) \) belongs to \( S_f \cap M \) (if a point is sensitive, so are all its iterates). Fix \( k \). Since \( S_f \cap M \) is countable by hypothesis and \( f^k \) is countably piecewise monotone, the set \( f^{-k}(S_f \cap M) \) has countably many components. Moreover, if \( K \) is a nondegenerate component of this set, then \( f^k(K) \) consists of exactly one point, which ensures that no point from \( \text{Int} K \) is sensitive. In particular, the set of sensitive points of \( B_f(M) \) that are mapped by \( f^k \) into \( S_f \cap M \) is countable for each given \( k \). We have thus demonstrated that \( S_f \cap J \) is countable, as desired. By induction we conclude that any atom of \( f \) contains at most countably many sensitive points.

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