THE ROBIN MEAN VALUE EQUATION I: A RANDOM WALK APPROACH TO THE THIRD BOUNDARY VALUE PROBLEM

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Abstract. We study the family of integral equations, called the Robin mean value equations (RMV), that are local averaged approximations to the Robin-Laplace boundary value problem (RL). When posed on $\mathcal{C}^{1,1}$-regular domains, we prove existence, uniqueness, equiboundedness and the comparison principle for solutions to (RMV). For the continuous right hand side of (RL), we show that solutions to (RMV) converge uniformly, in the limit of the vanishing radius of averaging, to the unique $W^{2,p}$ solution, which coincides with the unique viscosity solution of (RL). We further prove the lower bound on solutions to (RMV), which is consistent with the optimal lower bound for solutions to (RL). Our proofs employ martingale techniques, where (RMV) is interpreted as the dynamic programming principle along a suitable discrete stochastic process, interpolating between the reflecting and the stopped-at-exit Brownian walks.

1. Introduction

The purpose of this paper is to study the family of integral equations, parametrised by $\epsilon \to 0+$:

$$\text{(RMV)}_\epsilon \qquad u_\epsilon(x) = \big(1-\gamma s_\epsilon(x)\big) \fint_{B_\epsilon(x)\cap\mathcal{D}} u_\epsilon(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)} f(x) \qquad \text{for all } x\in\bar{\mathcal{D}},$$

posed on a bounded domain $\mathcal{D}\subset\mathbb{R}^N$, with a bounded, Borel function f, a positive constant γ, and s_ϵ appropriately given in (2.1). We call (RMV)_ϵ the *Robin mean value equation* and view it as the approximation to the *Robin-Laplace problem* (known as the *third boundary value problem*):

$$\text{(RL)} \qquad -\Delta u = f \quad \text{in } \mathcal{D}, \qquad \frac{\partial u}{\partial\vec{n}} + \gamma u = 0 \quad \text{on } \partial\mathcal{D}.$$

Our analysis of (RMV)_ϵ relies on its probabilistic interpretation as the dynamic programming principle along a discrete stochastic process $\{X_n^\epsilon\}_{n=0}^\infty$, which samples uniformly on the truncated balls $B_\epsilon(X_n^\epsilon)\cap\mathcal{D}$, and stops with probability $\gamma s_\epsilon(X_n^\epsilon)$ at each consecutive position X_n^ϵ. The process accumulates values of f until its stopping time τ^ϵ, whereas we define:

$$\text{(DPP)}_\epsilon \qquad \begin{aligned} u^\epsilon(x) = \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{n=0}^{\tau^{\epsilon,x}-1}\big(f\circ X_n^{\epsilon,x}\big)\Big] &= \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{n=0}^{\infty}\big(f\circ X_n^{\epsilon,x}\big)\Lambda_n^{\epsilon,x}\Big],\\ &\text{with } \Lambda_n^{\epsilon,x} = \prod_{i=1}^n\big(1-\gamma(s_\epsilon\circ X_{i-1}^{\epsilon,x})\big). \end{aligned}$$

The second representation above reflects the value of an infinite horison process, where the accumulation procedure never stops, but the consecutive evaluations of f are instead weighted by the probability of not having stopped so far.

Key words and phrases. Robin problem, third boundary value problem, oblique boundary conditions, dynamic programming principle, random walk, finite difference approximations, viscosity solutions.

In this paper we relate the three individual problems $(\mathrm{RMV})_\epsilon$, (RL) and $(\mathrm{DPP})_\epsilon$, combining the analytical and probabilistic techniques in their study. Our approach suggests how to view more general, nonlinear operators (like Δ_p, Δ_∞) subject to oblique-type boundary conditions, through their local averaged approximations of the type $(\mathrm{RMV})_\epsilon$. This approach has been previously successfully used in the Dirichlet and Neumann cases [26, 25, 21, 1, 8, 15].

We now summarize our main results. We work under the following basic hypotheses:

(BH) $\Big[$ The nonempty set $\mathcal{D}\subset\mathbb{R}^N$ is open, bounded, connected and of regularity $\mathcal{C}^{1,1}$. The function $f:\bar{\mathcal{D}}\to\mathbb{R}$ is bounded and Borel. The coefficient $\gamma>0$ is a positive constant.

Recall that $\mathcal{D}$ being $\mathcal{C}^{1,1}$ regular signifies that $\partial\mathcal{D}$ is locally a graph of a $\mathcal{C}^{1,1}$ function, which is equivalent to the uniform (two-sided) supporting sphere condition; see Lemma 2.2 for details.

Theorem 1.1. *Assume (BH) and let $\epsilon\ll 1$.*

(i) *Each problem $(\mathrm{RMV})_\epsilon$ has a unique solution $u_\epsilon=u^\epsilon$, coinciding with the value of $(\mathrm{DPP})_\epsilon$, that is Borel, bounded with a bound independent of ϵ, and obeys the comparison principle. For f is continuous / Hölder continuous / Lipschitz, u_ϵ inherits the same regularity.*

(ii) *When $f\in\mathcal{C}(\bar{\mathcal{D}})$, then $\{u_\epsilon\}_{\epsilon\to 0}$ converges uniformly on $\bar{\mathcal{D}}$ to $u\in\mathcal{C}(\bar{\mathcal{D}})$ that is the unique viscosity solution to (RL). In fact, u coincides with the unique $W^{2,p}(\mathcal{D})$ solution to (RL). Since the range of p covers $(1,\infty)$, it follows that $u\in\mathcal{C}^{1,\alpha}(\bar{\mathcal{D}})$ for any $\alpha\in(0,1)$.*

In a companion paper [17] we show that $\{u_\epsilon\}_{\epsilon\to 0}$ converges uniformly on $\bar{\mathcal{D}}$ to the unique $W^{2,p}(\mathcal{D})$ solution to (RL), for *any* bounded, Borel right hand side function f. To this end, in [17] we use probability techniques involving various couplings of random walks and yielding approximate Hölder regularity of u^ϵ in $(\mathrm{DPP})_\epsilon$ (Lipschitz in the interior and $\mathcal{C}^{0,\alpha}$ up to the boundary of $\mathcal{D}$, for any $\alpha\in(0,1)$). In the present paper, it suffices to combine the simpler martingale arguments close to the boundary of $\mathcal{D}$, with estimates on the Taylor remainder term for the Newtonian potential, via another probabilistic interpretation of u_ϵ, parallel to that in $(\mathrm{DPP})_\epsilon$.

By further martingale techniques we deduce the lower bound on u^ϵ in the general case of nonnegative bounded f, in function of γ and the radius of the inner supporting balls at $\partial\mathcal{D}$:

Theorem 1.2. *Assume (BH) with $f\geq 0$ and let $r>0$ satisfy:*

$$\text{for every } x\in\partial\mathcal{D} \text{ exists } B_r(a)\subset\mathcal{D} \quad \text{such that} \quad |x-a|=r.$$

Then the solution to $(\mathrm{RMV})_\epsilon$ obeys the following bound, for any radius $\bar{r}<r$ provided that $\epsilon\ll 1$:

$$u^\epsilon(x_0)\geq\frac{\bar{r}}{\gamma N}\cdot\inf_{\bar{\mathcal{D}}} f \qquad \text{for all } x_0\in\bar{\mathcal{D}}.$$

Clearly, uniform convergence of $\{u_\epsilon\}_{\epsilon\to 0}$ to u implies that $u\geq\frac{r}{\gamma N}\inf_{\bar{\mathcal{D}}} f$. This bound is optimal and equivalent to Theorem 1.2. For solutions to (RL), it may be obtained directly via the maximum principle.

1.1. **Prior results in related contexts.** To put $(\mathrm{RMV})_\epsilon$ in a more familiar setting, rewrite:

$$\begin{aligned} u_\epsilon(x) = p_\epsilon(x)\Big(&\fint_{B_\epsilon(x)}\mathbb{1}_{\mathcal{D}}(y)u_\epsilon(y)\;\mathrm{d}y+\frac{\epsilon^2}{2(N+2)}f(x)\Big)\\ &+\big(1-p_\epsilon(x)\big)\Big(\fint_{B_\epsilon(x)\cap\mathcal{D}}u_\epsilon(y)\;\mathrm{d}y+\frac{\epsilon^2}{2(N+2)}f(x)\Big), \qquad p_\epsilon(x)=\gamma s_\epsilon(x)\cdot\frac{|B_\epsilon(x)|}{|B_\epsilon(x)\setminus\mathcal{D}|}. \end{aligned}$$

Note that the interpolation weight p_ϵ is of order ϵ, for any $\gamma>0$. This rests in agreement with the fact that the Neumann averaging, corresponding to a higher order operator, must prevail. Indeed, we observe that the above formula interpolates between the dynamic programming principles for:

(i) the Dirichlet problem: $-\Delta u = f$ in $\mathcal{D}$, $u = 0$ on $\partial\mathcal{D}$, when $p_\epsilon = 1$. This heuristically corresponds to the limiting case $\gamma = \infty$. The probabilistic interpretation for the Dirichlet problem is classical and has been extensively studied in the literature both in the present linear case of Δ and the random walk [11], as well as in the nonlinear cases of Δ_p, $p \in (1,\infty]$ and the tug-of-war games with noise [25, 26, 21, 15].
In the continuous setting, the Perron solution to the homogeneous problem: $-\Delta u = 0$ in $\mathcal{D}$, $u = F$ on $\partial\mathcal{D}$ has the well known representation [11]:

$$u(x) = \mathbb{E}_x\big[F \circ \mathcal{B}_\tau\big] \doteq \mathbb{E}\big[F \circ (x + \mathcal{B}_{\tau_x})\big],$$

where $\{\mathcal{B}_t\}_{t\geq 0}$ denotes the standard N-dimensional Brownian motion, and where for each $x \in \mathcal{D}$ the stopping time τ_x is defined by: $\tau_x = \min\{t \geq 0;\ x + \mathcal{B}_t \in \partial\mathcal{D}\}$.

(ii) the Neumann problem: $-\Delta u = f$ in $\mathcal{D}$, $\frac{\partial u}{\partial \vec{n}} = 0$ on $\partial\mathcal{D}$, when $p_\epsilon = 0$. This corresponds to the case $\gamma = 0$ which is not covered in our paper. Indeed, the Neumann problem is not well posed unless $\int_\mathcal{D} f = 0$ and most of our proofs do not work in this limiting case. A related probabilistic approach [1] involves the reflected Brownian motion and the local boundary time, first introduced in [7] for $\mathcal{C}^3$ domains.
More precisely, in the continuous setting one defines the following family (nondecreasing in $t \geq 0$) of random variables, with the help of the reflected Brownian motion $\{\bar{\mathcal{B}}_t\}_{t\geq 0}$:

$$L_t = \lim_{\delta\to 0} \frac{1}{2\delta}\int_0^t \mathbb{1}_{\{\mathrm{dist}(\bar{\mathcal{B}}_s, \partial\mathcal{D}) < \delta\}}\ \mathrm{d}s.$$

Then, the solution to the homogeneous problem: $-\Delta u = 0$ in $\mathcal{D}$, $\frac{\partial u}{\partial \vec{n}} = F$ on $\partial\mathcal{D}$ satisfying $\int_{\partial\mathcal{D}} F = 0$, is given by the limiting expectation of the Lebesgue-Stieltjes integral in:

$$u(x) = \lim_{t\to\infty} \mathbb{E}_x\bigg[\int_0^t F \circ \bar{\mathcal{B}}_s\ \mathrm{d}L_s\bigg].$$

This approach was extended to $\mathcal{C}^{2,\alpha}$ domains [4] and to Lipschitz domains [3], together with a probabilistic proof of Hölder continuity in [3, Corollary 3.8].

We finally remark that the homogeneous Robin problem for the Laplace equation: $-\Delta u = 0$ in $\mathcal{D}$, $\frac{\partial u}{\partial \vec{n}} + \gamma u = F$ on $\partial\mathcal{D}$, was first studied probabilistically in [24] for $\mathcal{C}^3$ domains. The corresponding representation via the boundary local time of reflected Brownian motion was shown to be:

$$u(x) = \lim_{t\to\infty} \mathbb{E}_x\bigg[\int_0^t e^{-\gamma L_s}(F \circ \bar{\mathcal{B}}_s)\ \mathrm{d}L_s\bigg] = \mathbb{E}_x\bigg[\int_0^\infty e^{-\gamma L_s}(F \circ \bar{\mathcal{B}}_s)\ \mathrm{d}L_s\bigg].$$

Observe that the coefficient $e^{-\gamma L_s}$ above plays the same role as the factor $\Lambda^{\epsilon,x}_{\lfloor s/n\rfloor}$ in our discrete representation $(\mathrm{DPP})_\epsilon$. The same problem was studied in certain fractal domains in [2]. The mixed Dirichlet-Neumann problem for Δ_∞ in the context of the game from [25], under the assumption of $\mathcal{C}^1$ regular Neumann boundary and Lipschitz Dirichlet data, was addressed in [8].

1.2. **Relation between $(\mathrm{RMV})_\epsilon$ and (RL).** To motivate the role of the coefficient $s_\epsilon(x)$ that will be precisely defined in (2.1), we average on the truncated ball $B_\epsilon(x)\cap\mathcal{D}$ the Taylor expansion:

$$u(y) = u(x) + \langle\nabla u(x), y - x\rangle + \frac{1}{2}\langle\nabla^2 u(x) : (y-x)^{\otimes 2}\rangle + o(|y-x|^2)$$

When $d = \mathrm{dist}(x, \partial\mathcal{D}) \geq \epsilon$, this procedure leads to the familiar formula:

$$u(x) = \fint_{B_\epsilon(x)} u(y)\ \mathrm{d}y - \frac{\epsilon^2}{2(N+2)}\Delta u(x) + o(\epsilon^2),$$

whose leading order terms readily coincide with $(\mathrm{RMV})_\epsilon$ after replacing $-\Delta u$ with f and after setting $s_\epsilon(x) = 0$. In case of $d < \epsilon$ when $x \approx \bar x \in \partial\mathcal{D}$, the same reasoning requires calculating the possibly nonzero average $\fint_{B_\epsilon(x)\cap\mathcal{D}} y - x \;\mathrm{d}y$. For sufficiently regular domains $\mathcal{D}$, one can approximate this term by the average on the ball $B_\epsilon(x)$ truncated with the tangent plane to $\partial\mathcal{D}$ at $\bar x$, rather than by the surface $\partial\mathcal{D}$. This simpler average may be then directly computed as: $-s_\epsilon(x)\vec n(\bar x) \sim -\epsilon\big(1 - \big(\frac{d}{\epsilon}\big)^2\big)^{\frac{N+1}{2}}\vec n(\bar x)$. Assuming the boundary condition $u(\bar x) + \gamma\frac{\partial u}{\partial \vec n}(\bar x) = 0$, the first two terms of the discussed averaged Taylor expansion thus become:

$$u(x) - \big\langle \nabla u(x), s_\epsilon(x)\vec n(\bar x)\big\rangle = u(x) - s_\epsilon(x)\frac{\partial u}{\partial \vec n}(\bar x) + O(\epsilon s_\epsilon(x)) = u(x) + \gamma s_\epsilon(x)u(x) + O(\epsilon s_\epsilon(x)).$$

Since $(1+\gamma s_\epsilon)^{-1} = (1-\gamma s_\epsilon) + O(s_\epsilon^2)$, we conclude that:

$$u(x) = (1-\gamma s_\epsilon(x))\fint_{B_\epsilon(x)} u(y)\;\mathrm{d}y - \frac{\epsilon^2}{2(N+2)}\Delta u(x) + O(\epsilon s_\epsilon(x)) + o(\epsilon^2), \tag{1.1}$$

which coincides with $(\mathrm{RMV})_\epsilon$ at its leading order terms.

We remark that the Robin problem (RL), called also the third boundary value problem / impedance boundary problem / convective boundary problem, has received attention due to its applications in many contexts in science and engineering. Using classical Schauder estimates, it follows [12, Chapter 6.7] that on a bounded domain $\mathcal{D}$ of regularity $\mathcal{C}^{2,\alpha}$, the general strictly elliptic problem $Lu = f$ with $\mathcal{C}^\alpha(\bar{\mathcal{D}})$-regular coefficients and $f \in \mathcal{C}^\alpha(\bar{\mathcal{D}})$, subject to the oblique boundary conditions: $\langle\beta(x), \nabla u(x)\rangle + \gamma(x)u(x) = \phi(x)$ posed with $\gamma, \beta, \phi \in \mathcal{C}^{1,\alpha}(\partial D)$ where $\gamma\langle\beta, \vec n\rangle > 0$, has a unique solution $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$, that satisfies the usual a-priori bounds.

Much of the modern theory for nonlinear boundary value problems modeled on (RL), is contained in the recent extensive monograph [18]. It is shown in Theorem 1.26 there, that solutions to linear oblique problems in Lipschitz domains are Hölder continuous. Further, in Theorem 4.40 and Corollary 4.41 it is shown that regularity $\mathcal{C}^{1,\alpha}$ of $\mathcal{D}$ suffices for the solution regularity $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$, provided that $f \in \mathcal{C}^\alpha(\bar{\mathcal{D}})$ and $\beta \in \mathcal{C}^{1,\alpha}(\partial\mathcal{D})$. We observe that for (RL), the obliqueness vector $\beta = \vec n$ is only Lipschitz and thus one cannot, in general, expect that $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$. For example, when $N = 2$ then in local coordinates $\partial\mathcal{D}$ may be parametrised as the graph $\{(x_1, \phi(x_1))\}$ of some function $\phi \in \mathcal{C}^{1,1}$. Writing $\vec n = (\phi'(x_1), -1)/\sqrt{1+|\phi'(x_1)|^2}$, it is not hard to check that if $u \in \mathcal{C}^{2,\alpha}(\bar{\mathcal{D}})$, then the boundary condition in (RL) implies that ϕ' is a solution to a quadratic equation with $\mathcal{C}^{1,\alpha}$ coefficients, and thus it must automatically be $\mathcal{C}^{1,\alpha}$.

In [18, Theorem 6.30] it is proved that if $\mathcal{D}$ has regularity $\mathcal{C}^{1,\alpha}$ and $f \in L^p(\mathcal{D})$ with $p \in (1, \frac{1}{1-\alpha})$ then $u \in W^{2,p}(\mathcal{D})$ in (RL). Consequently, when $f \in L^\infty$, we get that $u \in \mathcal{C}^{1,\alpha}(\bar{\mathcal{D}})$ for any $\alpha \in (0,1)$. Analysis of (RL) in non-smooth domains, including sets with a rectifiable topological boundary having finite $(N-1)$-dimensional Hausdorff measure, can be found in [23, 11, 6].

1.3. **The structure of this paper.** In section 2 we prove Theorem 1.1 (i) together with some preliminary geometric lemmas regarding s_ϵ. In section 3 we derive the expansion (1.1) for $\mathcal{C}^2(\bar{\mathcal{D}})$ solutions u to (RL) and show the uniform convergence of $\{u_\epsilon\}_{\epsilon\to 0}$ to u in this restricted setting. In section 4 we recall the definition of viscosity solutions to (RL) when $f \in \mathcal{C}(\bar{\mathcal{D}})$ and prove in Theorem 4.2, that any uniform limit of $\{u_\epsilon\}_{\epsilon\to 0}$ must automatically be a viscosity solution. In fact, viscosity solutions to (RL) are unique under the uniform supporting spheres assumption. For completeness, we reproduce the proof of this folklore statement in Lemma 9.2 in section 9.

In section 5 we develop the probability setup related to $(\mathrm{RMV})_\epsilon$, where $u_\epsilon = u^\epsilon$. We define the stochastic process $\{X_n^\epsilon\}_{n=0}^\infty$ in $(\mathrm{DPP})_\epsilon$ by (5.2) and sketch two procedures achieving the uniform distribution of its positions on the truncated balls $B_\epsilon(X_n^\epsilon)\cap\mathcal{D}$: via the rejection sampling, and via the Knothe-Rosenblatt rearrangement. The first easy bound on the expectation of the stopping time τ^ϵ, of the order C/ϵ^2, is then refined in section 8 where we prove Theorem 8.2 that serves to deduce Theorem 1.2. As a byproduct, we obtain the lower bound on the limit u in Theorem 1.1 (ii), which is independently derived in Lemma 8.4 for $u \in \mathcal{C}^1(\bar{\mathcal{D}})$ satisfying (RL).

The second probabilistic interpretation of u_ϵ is proposed in section 6: instead of stopping the accumulation of values f along the process $\{X_n^\epsilon\}_{n=0}^\infty$ at τ^ϵ in $(\mathrm{DPP})_\epsilon$, the accumulation proceeds indefinitely but each $f(X_n)$ is weighted by the product of Dirichlet factors $\Lambda_n = \prod_{i=0}^{n-1}(1-\gamma s_\epsilon(X_i))$. The main convergence statement in Theorem 1.1 (ii) is then proved in section 7. We first estimate the remainder term in the Robin expansion (1.1) for u that is a $W^{2,p}$ solution of (RL). The estimate at the boundary of $\mathcal{D}$ follows from an estimate on $\sum_{n=0}^\infty \Lambda_n$ in Lemma 6.2. The estimate away from $\partial\mathcal{D}$ is given in Theorem 7.1: the said remainder equals the difference of f from its convolution with a suitable probabilistic kernel. This suffices to pass to the limit when $f \in \mathcal{C}(\bar{\mathcal{D}})$; for a bounded f that is only Borel, the same argument is refined in the companion paper [17].

1.4. **Notation.** Given a $\mathcal{C}^1$ domain $\mathcal{D} \subset \mathbb{R}^N$, we denote by $\vec{n}(x)$ the outward unit normal vector at $x \in \partial\mathcal{D}$, and by $\pi_{\partial\mathcal{D}}x$ the projection onto $\partial\mathcal{D}$ along the normal $\vec{n}(\pi_{\partial\mathcal{D}}x)$, defined for each $x \in \bar{\mathcal{D}}$ with sufficiently small distance from $\partial\mathcal{D}$. Unless specified otherwise, by C we denote any universal positive constant that may depend on $\mathcal{D}$, γ and f, but not on ϵ, x or other parameter quantities. The Landau symbols $\mathcal{O}$ and o likewise have the same uniformity properties. By $\epsilon \ll 1$ and $C \gg 1$ we mean a "sufficiently small" and a "sufficiently large" positive number.

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2. The Robin mean value equation and some geometrical lemmas

In this section, we prove Theorem 1.1 (i). Recall that we work under the basic hypotheses (BH) and that we are concerned with the family of equations, parametrised by $\epsilon > 0$, given by:

$$(\mathrm{RMV})_\epsilon \qquad u_\epsilon(x) = \big(1-\gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} u_\epsilon(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)}f(x) \qquad \text{for all } x \in \bar{\mathcal{D}}.$$

To define the coefficient $s_\epsilon(x)$ above, we introduce the following notation (see Figure 2.1):

$$d_\epsilon(x) = \min\big\{1, \frac{1}{\epsilon}\mathrm{dist}(x,\partial\mathcal{D})\big\} \in [0,1] \qquad \text{for all } x \in \mathcal{D},\ \epsilon > 0,$$
$$B_1^k = B_1(0) \subset \mathbb{R}^k, \qquad B_{1,d}^k = B_1^k \cap \{y_k < d\} \qquad \text{for all } \ d \in [0,1].$$

Then we set:

$$s_\epsilon(x) = \frac{|B_1^{N-1}|}{(N+1)|B_{1,d_\epsilon(x)}^N|}\cdot \epsilon\big(1-d_\epsilon(x)^2\big)^{\frac{N+1}{2}} \qquad \text{for all } x \in \bar{\mathcal{D}},\ \epsilon > 0. \tag{2.1}$$

In order to prove existence and uniqueness of solutions to $(\mathrm{RMV})_\epsilon$, we need some geometrical lemmas regarding the quantity s_ϵ.

Lemma 2.1. *Assume that $B_\epsilon(x)\cap\mathcal{D} = x + \epsilon B_{1,d_\epsilon(x)}^N$. Then:*

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} y - x\;\mathrm{d}y = -s_\epsilon(x)e_N.$$

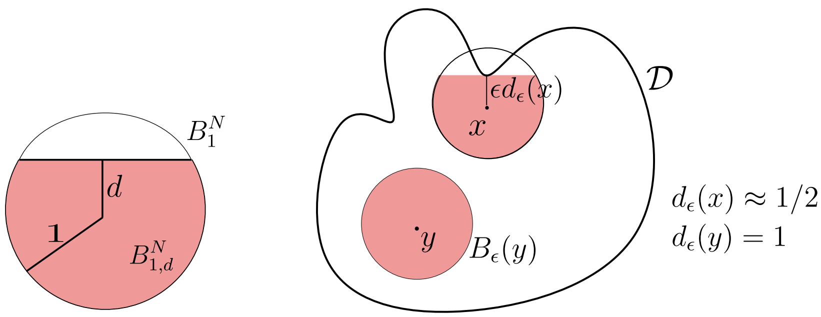

FIGURE 2.1. The referential truncated ball and the scaled distances from $\partial\mathcal{D}$.

Proof. The result follows by an explicit calculation:

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} y - x \,\mathrm{d}y = \epsilon\Big(\fint_{B^N_{1,d_\epsilon(x)}} y_N \,\mathrm{d}y\Big)e_N = \epsilon\frac{|B_1^{N-1}|}{|B^N_{1,d_\epsilon(x)}|}\Big(\int_{-1}^{d_\epsilon(x)} s(1-s^2)^{\frac{N-1}{2}}\,\mathrm{d}s\Big)e_N$$
$$= -\epsilon\frac{|B_1^{N-1}|}{(N+1)|B^N_{1,d_\epsilon(x)}|}\big(1-d_\epsilon(x)^2\big)^{\frac{N+1}{2}}e_N = -s_\epsilon(x)e_N,$$

where we applied change of variables and the fact that $\fint_{B^N_{1,d}} y_i \,\mathrm{d}y = 0$ for all $i \neq N$. ∎

We now extend Lemma 2.1 to domains with curved boundaries. Recall that $\mathcal{D}$ is said to be of class $\mathcal{C}^{1,1}$ provided that $\partial\mathcal{D}$ is locally a graph of a $\mathcal{C}^{1,1}$ function. Equivalently, such $\mathcal{D}$ satisfies the uniform (two-sided) supporting sphere condition, stated below. This result has been first shown in [20, Section 2]; we refer to [16] for a self-contained discussion and an elementary proof.

Lemma 2.2. *An open, bounded set $\mathcal{D} \subset \mathbb{R}^N$ is of class $\mathcal{C}^{1,1}$ if and only if there exists a radius $r > 0$ such that for every $x \in \partial\mathcal{D}$ there exist $a, b \in \mathbb{R}^N$ satisfying:*

$$B_r(a) \subset \mathcal{D}, \qquad B_r(b) \subset \mathbb{R}^N \setminus \bar{\mathcal{D}} \quad \textit{and} \quad |x-a| = |x-b| = r.$$

Moreover, the global Lipschitz constant of $\vec{n}$ can be taken as the inverse of the supporting radius:

$$|\vec{n}(x) - \vec{n}(y)| \leq \frac{1}{r}|x-y| \qquad \textit{for all } x, y \in \partial\mathcal{D}.$$

Lemma 2.3. *Let $\mathcal{D}$ be as in (BH). Then, for all $\epsilon \ll 1$ and all $x \in \bar{\mathcal{D}}$ we have:*

(i) $\dfrac{|B_\epsilon(x) \setminus \mathcal{D}|}{|B_\epsilon(0)|} \leq C\dfrac{s_\epsilon(x)}{\epsilon}$.

(ii) *We bound the volume of the following symmetric difference:*

$$\frac{\big|\big(B_\epsilon(x)\cap\mathcal{D}\big)\triangle\big(x + \epsilon R_x B^N_{1,d_\epsilon(x)}\big)\big|}{\big|B_\epsilon(0)\big|} \leq Cs_\epsilon(x) \qquad \textit{for all } x \in \bar{\mathcal{D}}, \tag{2.2}$$

where for $\mathrm{dist}(x, \partial\mathcal{D}) < \epsilon$ *we set $R_x \in SO(N)$ to be a rotation satisfying $R_x e_N = \vec{n}\big(\pi_{\partial\mathcal{D}}(x)\big)$, whereas for* $\mathrm{dist}(x,\partial\mathcal{D}) \geq \epsilon$ *both sides of (2.2) are null for any $R_x \in SO(N)$.*

(iii) *There holds:*

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} y - x \,\mathrm{d}y = -s_\epsilon(x)\vec{n}\big(\pi_{\partial\mathcal{D}}x\big) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) \qquad \textit{for all } x \in \bar{\mathcal{D}}. \tag{2.3}$$

Proof. **1.** Let $r > 0$ be the radius of the supporting spheres as in Lemma 2.2. It suffices to treat the case in which $\mathrm{dist}(x, \partial\mathcal{D}) < \epsilon \ll r$ and $\vec{n}(\pi_{\partial\mathcal{D}}x) = e_N$. We first prove (ii). Since $\partial\mathcal{D}$ is contained in the region between the two supporting spheres, the quantity in (2.2) is bounded by:

$$\begin{split} & \frac{\Big|B_\epsilon(x)\setminus\Big(B_r\big(x+(\epsilon d_\epsilon(x)-r)e_N\big)\cup B_r\big(x+(\epsilon d_\epsilon(x)+r)e_N\big)\Big)\Big|}{\big|B_\epsilon(0)\big|} \\ & \qquad \leq \frac{2\cdot\Big|\epsilon B^N_{1,d_\epsilon(x)}\setminus B_r\big((\epsilon d_\epsilon(x)-r)e_N\big)\Big|}{\big|B_\epsilon(0)\big|} \leq C\Big|B^N_{1,d_\epsilon(x)}\setminus B_{r/\epsilon}\big((d_\epsilon(x)-\frac{r}{\epsilon})e_N\big)\Big|. \end{split}$$

Writing $d = d_\epsilon(x)$ and $\bar{r} = \frac{r}{\epsilon}$, the measure above is further bounded by the following quantity:

$$q_{d,\bar{r}} = \int_{\frac{2\bar{r}d-1-d^2}{2(\bar{r}-d)}}^{d} (1-s^2)^{\frac{N-1}{2}}\,\mathrm{d}s. \tag{2.4}$$

Observe that for $\bar{r} \geq 2$ we have: $d - \frac{2\bar{r}d-1-d^2}{2(\bar{r}-d)} = \frac{1-d^2}{2(\bar{r}-d)} \leq \frac{1}{2(\bar{r}-1)} \leq \frac{1}{\bar{r}} = \frac{\epsilon}{r}$, so automatically:

$$q_{d,\bar{r}} \leq C\epsilon \leq C s_\epsilon(x) \qquad \text{for all } \ d\in[0,\frac{1}{2}].$$

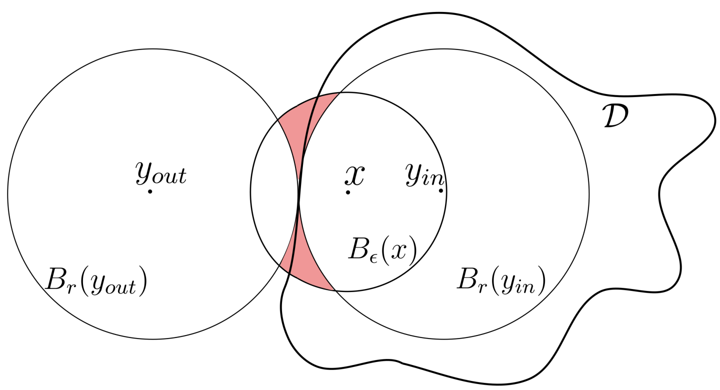

FIGURE 2.2. The region estimated in the proof of Lemma 2.3 (ii).

On the other hand, for $d \in (\frac{1}{2}, 1]$ and $\bar{r} \geq 2$, the lower integration limit in (2.4) is nonnegative, and thus we may use the fact that the function $s \mapsto (1-s^2)^{\frac{N-1}{2}}$ is decreasing on $[0,1]$, to obtain:

$$\begin{split} q_{d,\bar{r}} & \leq C\Big(d - \frac{2\bar{r}d-1-d^2}{2(\bar{r}-d)}\Big)\Big(1-\Big(\frac{2\bar{r}d-1-d^2}{2(\bar{r}-d)}\Big)^2\Big)^{\frac{N-1}{2}} \\ & \leq C\frac{1-d^2}{\bar{r}}(1-d^2)^{\frac{N-1}{2}} = C\epsilon\frac{1-d^2}{\bar{r}}(1-d^2)^{\frac{N+1}{2}} = Cs_\epsilon(x) \qquad \text{for all } \ d\in(\frac{1}{2},1], \end{split}$$

in virtue of the simple estimate:

$$1-\Big(\frac{2\bar{r}d-1-d^2}{2(\bar{r}-d)}\Big)^2 \leq 1-\Big(d-\frac{1-d^2}{\bar{r}}\Big)^2 \leq (1-d^2)\big(1+\frac{2}{\bar{r}}\big).$$

This concludes the proof of (2.2), for all $\epsilon \ll 1$ that guarantee $\bar{r} \geq 2$.

2. The bound (2.3) in (iii) is implied directly by (2.2) and Lemma 2.1. To show (i), we estimate:

$$\begin{aligned}\frac{|B_\epsilon(x)\setminus\mathcal{D}|}{|B_\epsilon(0)|} &\leq \frac{|B_1^N\setminus B_{1,d}^N|}{|B_1^N|} + q_{d,\bar r} \leq C\int_d^1 (1-s^2)^{\frac{N-1}{2}}\,\mathrm{d}s + q_{d,\bar r}\\ &\leq C\big(1-d^2\big)^{\frac{N+1}{2}} + q_{d,\bar r} \leq C\frac{s_\epsilon(x)}{\epsilon} + Cs_\epsilon(x),\end{aligned}$$

which ends the proof. ∎

The next result provides the second order counterpart of the first order bound (2.3):

Lemma 2.4. *Let $\mathcal{D}$ be as in (BH). Then:*

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} (y-x)^{\otimes 2}\,\mathrm{d}y = \frac{\epsilon^2}{N+2}Id_N + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) \qquad \textit{for all } x\in\bar{\mathcal{D}}. \tag{2.5}$$

Proof. We first assume that $B_\epsilon(x)\cap\mathcal{D} = x+\epsilon B^N_{1,d_\epsilon(x)}$. By a change of variables:

$$\begin{aligned}\fint_{B_\epsilon(x)\cap\mathcal{D}} (y-x)^{\otimes 2}\,\mathrm{d}y &= \epsilon^2\fint_{B^N_{1,d_\epsilon(x)}} y^{\otimes 2}\,\mathrm{d}y = \epsilon^2\fint_{B_1^N} y^{\otimes 2}\,\mathrm{d}y + \epsilon^2\mathcal{O}\big(|B_1^N\setminus B^N_{1,d_\epsilon(x)}|\big)\\ &= \frac{\epsilon^2}{N+2}Id_N + \mathcal{O}\big(\epsilon s_\epsilon(x)\big),\end{aligned}$$

where we have used that: $\fint_{B_1^N} y^{\otimes 2}\,\mathrm{d}y = \big(\fint_{B_1^N} y_1^2\,\mathrm{d}y\big)Id_N = \frac{1}{N+2}Id_N$, and:

$$|B_1^N\setminus B^N_{1,d_\epsilon(x)}| = |B_1^{N-1}|\int_{d_\epsilon(x)}^1 (1-s^2)^{\frac{N-1}{2}}\,\mathrm{d}s \leq C\big(1-d_\epsilon(x)^2\big)^{\frac{N+1}{2}} = C\frac{s_\epsilon(x)}{\epsilon}.$$

In the general case, we apply the bound (2.2) which introduces an additional error term of order $\mathcal{O}(\epsilon^2 s_\epsilon(x)) \leq \mathcal{O}(\epsilon s_\epsilon(x))$. This ends the proof. ∎

We are now ready for the main statement of this section:

Theorem 2.5. *Assume (BH). For every $\epsilon \ll 1$, the problem (RMV)$_\epsilon$ has a unique solution $u_\epsilon = u_\epsilon^f$ that is bounded and Borel. These solutions obey comparison principle, in the sense that $f\leq g$ implies $u_\epsilon^f \leq u_\epsilon^g$ in $\bar{\mathcal{D}}$. The family $\{u_\epsilon^f\}_{\epsilon\to 0}$ is equibounded, with a bound independent of ϵ.*

Proof. **1.** For every bounded, Borel $u:\bar{\mathcal{D}}\to\mathbb{R}$, define $T_\epsilon^f u:\bar{\mathcal{D}}\to\mathbb{R}$ by:

$$T_\epsilon^f u(x) = \big(1-\gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\bar{\mathcal{D}}} u(y)\,\mathrm{d}y + \frac{\epsilon^2}{2(N+2)}f(x) \qquad \text{for all } x\in\bar{\mathcal{D}}.$$

Clearly, the operator T_ϵ^f is monotone, in the sense that $u\leq v$ implies $T_\epsilon^f(u)\leq T_\epsilon^f(v)$ in $\bar{\mathcal{D}}$. Assume first that $f\geq 0$ and recursively define the sequence of Borel functions $\{u_n\}_{n=1}^\infty$:

$$u_0\equiv 0, \qquad u_n = (T_\epsilon^f)^n u_0.$$

Since $u_1\geq 0$, it follows that $\{u_n\}_{n=1}^\infty$ is pointwise increasing. Below, we will show that it is uniformly bounded. This property will automatically result in the pointwise convergence to some bounded, nonnegative, Borel limit function $u_\epsilon:\bar{\mathcal{D}}\to\mathbb{R}$. Applying the monotone convergence theorem, we get that $u_{n+1} = T_\epsilon^f u_n$ converges to $T_\epsilon^f u_\epsilon$ and therefore $u_\epsilon = T_\epsilon^f u_\epsilon$ solves (RMV)$_\epsilon$. For a sign changing right hand side f, we decompose: $f = f^+ - f^-$ with $f^+, f^-\geq 0$ and observe that $u_\epsilon^f = u_\epsilon^{f^+} - u_\epsilon^{f^-}$ solves (RMV)$_\epsilon$ in this general case as well.

2. To achieve the claimed boundedness, we will now prove that there exist constants $C_1, C_2 \gg 1$ such that defining a quadratic function $v(x) = C_1 - C_2|x|^2$, there holds:

$$u \le v \quad \text{implies} \quad T_\epsilon^f u \le v \quad \text{in } \bar{\mathcal{D}}, \tag{2.6}$$

for every bounded, Borel $u : \bar{\mathcal{D}} \to \mathbb{R}$ and every $\epsilon \ll 1$. We start by noting that the monotonicity of T_ϵ^f yields, when C_1 is much larger than C_2:

$$\begin{aligned}
\big(T_\epsilon^f u - v\big)(x) &\le \big(T_\epsilon^f v - v\big)(x) \\
&= \fint_{B_\epsilon(x)\cap\mathcal{D}} v(y) - v(x)\;\mathrm{d}y - \gamma s_\epsilon(x) \fint_{B_\epsilon(x)\cap\mathcal{D}} v(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)} f(x) \\
&\le -C_2 \fint_{B_\epsilon(x)\cap\mathcal{D}} |y|^2 - |x|^2\;\mathrm{d}y - \frac{C_1}{2} s_\epsilon(x) + \mathcal{O}(\epsilon^2) \\
&\le -2C_2 \Big\langle \fint_{B_\epsilon(x)\cap\mathcal{D}} y - x\;\mathrm{d}y, x\Big\rangle - \frac{C_1}{2} s_\epsilon(x) - C_2 \fint_{B_\epsilon(x)\cap\mathcal{D}} |y-x|^2\;\mathrm{d}y + \mathcal{O}(\epsilon^2),
\end{aligned}$$

Recalling (2.3) and (2.5) implies that the above quantity is nonpositive, provided that we choose $C_1 \gg C_2 \gg 1$. This proves (2.6).

3. We thus conclude existence of solutions to $(\mathrm{RMV})_\epsilon$ and the stated comparison principle, which follows by construction. It remains to show uniqueness; to this end, consider the difference $u = u_{\epsilon,1} - u_{\epsilon,2}$ of some two solutions $u_{\epsilon,1}$ and $u_{\epsilon,2}$, to $(\mathrm{RMV})_\epsilon$. Since:

$$u(x) = (1 - \gamma s_\epsilon(x)) \fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\;\mathrm{d}y \qquad \text{for all } x \in \bar{\mathcal{D}},$$

it follows that $u \in \mathcal{C}(\bar{\mathcal{D}})$. Assume, by contradiction, that u attains a positive maximum at some $x_0 \in \bar{\mathcal{D}}$. Then, there must be: $u(x_0) \le (1 - \gamma s_\epsilon(x_0))u(x_0)$, implying that: $s_\epsilon(x_0) = 0$. Consequently:

$$u(x_0) = \fint_{B_\epsilon(x_0)} u(y)\;\mathrm{d}y \le u(x_0)$$

so $u = u(x_0)$ is constant in $B_\epsilon(x_0)$. Repeating this argument, we obtain that u attains its maximum at some $\bar{x}_0 \in \partial\mathcal{D} + B_\epsilon(0)$. This contradicts $s_\epsilon(\bar{x}_0) = 0$ and ends the proof of Theorem 2.5. ∎

Corollary 2.6. *In the context of Theorem 2.5, continuity, Hölder continuity or Lipschitz continuity of f on $\bar{\mathcal{D}}$ implies, respectively: continuity, Hölder continuity, or Lipschitz continuity of the unique solution u_ϵ^f to $(\mathrm{RMV})_\epsilon$.*

Proof. Observe that the map s_ϵ is Lipschitz continuous on $\bar{\mathcal{D}}$ as the smooth function of the scaled distance d_ϵ, and $\bar{\mathcal{D}} \ni x \mapsto \int_{B_\epsilon(x)\cap\mathcal{D}} u(y)\;\mathrm{d}y$ is Lipschitz as well in view of the boundedness of u. Similarly, the map $\bar{\mathcal{D}} \ni x \mapsto |B_\epsilon(x)\cap\mathcal{D}|$ is Lipschitz. Thus, the first term in the right hand side of $(\mathrm{RMV})_\epsilon$ is Lipschitz continuous, which proves the claim. ∎

3. The Taylor expansion and convergence to classical solutions

In this section, we show the expansion (1.1) and prove convergence statement in Theorem 1.1 (ii) under higher regularity assumptions, pertaining to classical solutions of (RL).

Theorem 3.1. *Assume (BH). Let $u \in \mathcal{C}^2(\bar{\mathcal{D}})$, $f \in \mathcal{C}(\bar{\mathcal{D}})$ satisfy (RL). Then we have, as $\epsilon \to 0$:*

$$u(x) = \big(1 - \gamma s_\epsilon(x)\big) \fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\;\mathrm{d}y + \frac{\epsilon^2}{2(N+2)} f(x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) + o(\epsilon^2) \qquad \textit{for all } x \in \bar{\mathcal{D}},$$

where the Landau symbols $\mathcal{O}$ and o are uniform in x, but may depend on u, f, $\mathcal{D}$ and γ.

Proof. Without loss of generality, we may take $u \in \mathcal{C}^2(\mathbb{R}^N)$, so that the Taylor expansion of u results in the following expansion of the average, uniformly in $x \in \bar{\mathcal{D}}$:

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\,\mathrm{d}y = u(x) + \big\langle \nabla u(x), \fint_{B_\epsilon(x)\cap\mathcal{D}} y - x\,\mathrm{d}y\big\rangle + \frac{1}{2}\big\langle \nabla^2 u(x) : \fint_{B_\epsilon(x)\cap\mathcal{D}} (y-x)^{\otimes 2}\,\mathrm{d}y\big\rangle + o(\epsilon^2).$$

Recalling (2.3) and using boundary condition in (RL), the linear term above becomes:

$$\begin{aligned}\big\langle \nabla u(x), \fint_{B_\epsilon(x)\cap\mathcal{D}} y - x\,\mathrm{d}y\big\rangle &= -s_\epsilon(x)\big\langle \nabla u(x), \vec{n}\big(\pi_{\partial\mathcal{D}}x\big)\big\rangle + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) \\ &= -s_\epsilon(x)\frac{\partial u}{\partial \vec{n}}(\pi_{\partial\mathcal{D}}x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) = \gamma s_\epsilon(x) u(\pi_{\partial\mathcal{D}}x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) \\ &= \gamma s_\epsilon(x) u(x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big).\end{aligned}$$

By (2.5) and (RL), the quadratic term becomes:

$$\begin{aligned}\big\langle \nabla^2 u(x) : \fint_{B_\epsilon(x)\cap\mathcal{D}} (y-x)^{\otimes 2}\,\mathrm{d}y\big\rangle &= \frac{\epsilon^2}{N+2}\big\langle \nabla^2 u(x) : Id_N\big\rangle + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) \\ &= \frac{\epsilon^2}{N+2}\Delta u(x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) = -\frac{\epsilon^2}{N+2} f(x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big).\end{aligned}$$

In conclusion, it follows that:

$$\fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\,\mathrm{d}y = \big(1 + \gamma s_\epsilon(x)\big) u(x) - \frac{\epsilon^2}{2(N+2)} f(x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big) + o(\epsilon^2). \tag{3.1}$$

Observing that:

$$\frac{1}{1+\gamma s_\epsilon(x)} = 1 - \gamma s_\epsilon(x) + \mathcal{O}\big(s_\epsilon(x)^2\big) = 1 - \gamma s_\epsilon(x) + \mathcal{O}\big(\epsilon s_\epsilon(x)\big)$$

and dividing (3.1) by $1 + \gamma s_\epsilon(x)$, yields the claim ■

The statement below is shown using arguments similar to the proof of Theorem 2.5.

Theorem 3.2. *Under the assumptions of Theorem 3.1, the sequence of (unique) solutions $\{u_\epsilon\}_{\epsilon\to 0}$ to $(\mathrm{RMV})_\epsilon$ converges to u, uniformly on $\bar{\mathcal{D}}$.*

Proof. **1.** Define the sequence of positive numbers $\{a_\epsilon\}_{\epsilon\to 0}$, converging to 0 and such that the second error term in Taylor's expansion in Theorem 3.1 satisfies: $o(\epsilon^2) \leq a_\epsilon \epsilon^2$, uniformly in $x \in \bar{\mathcal{D}}$. For two parameters: a sufficiently small $\delta > 0$ and a large $C > \frac{N+2}{N\delta}$, define the quadratic functions:

$$v_\epsilon(x) = C\big(a_\epsilon(1 - \delta|x|^2) + \epsilon\big).$$

Clearly, $v_\epsilon > 0$ on $\bar{\mathcal{D}}$ if $\delta \ll 1$. Then we also have:

$$\begin{aligned}&v_\epsilon(x) - \big(1 - \gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} v_\epsilon(y)\,\mathrm{d}y \\ &\quad= Ca_\epsilon\Big(1 - \delta|x|^2 - (1 - \gamma s_\epsilon(x)) + \delta(1 - \gamma s_\epsilon(x))\fint_{B_\epsilon(x)\cap\mathcal{D}} |y|^2\,\mathrm{d}y\Big) + C\epsilon s_\epsilon(x) \\ &\quad= Ca_\epsilon\Big(s_\epsilon(x)\big(1 - \delta\fint_{B_\epsilon(x)\cap\mathcal{D}} |y|^2\,\mathrm{d}y\big) + \delta\fint_{B_\epsilon(x)\cap\mathcal{D}} |y|^2 - |x|^2\,\mathrm{d}y\Big) + C\epsilon s_\epsilon(x).\end{aligned}$$

Writing $\fint_{B_\epsilon(x)\cap\mathcal{D}} |y|^2 - |x|^2 \,\mathrm{d}y = \fint_{B_\epsilon(x)\cap\mathcal{D}} |y-x|^2 \,\mathrm{d}y + 2\big\langle \fint_{B_\epsilon(x)\cap\mathcal{D}} y - x \,\mathrm{d}y, x\big\rangle$ and applying (2.3) and (2.5), the above quantity is bounded from below, again when $\delta \ll 1$, by:

$$Ca_\epsilon\Big(\frac{\gamma}{2}s_\epsilon(x) + \delta\Big(\frac{\epsilon^2 N}{N+2} + \mathcal{O}(s_\epsilon(x))\Big)\Big) + C\epsilon s_\epsilon(x) \geq Ca_\epsilon\delta\frac{\epsilon^2 N}{N+2} + C\epsilon s_\epsilon(x) \geq a_\epsilon\epsilon^2 + C\epsilon s_\epsilon(x),$$

so that:

$$v_\epsilon(x) - \big(1 - \gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} v_\epsilon(y)\,\mathrm{d}y \geq a_\epsilon\epsilon^2 + C\epsilon s_\epsilon(x). \tag{3.2}$$

2. Consider the difference:

$$w_\epsilon = u - u_\epsilon - v_\epsilon.$$

By (3.2), the expansion in Lemma 3.1, and by the assumed identities $(\mathrm{RMV})_\epsilon$, we get:

$$w_\epsilon(x) - \big(1 - \gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} w_\epsilon(y)\,\mathrm{d}y \leq \mathcal{O}(\epsilon s_\epsilon(x)) + a_\epsilon\epsilon^2 - \big(a_\epsilon\epsilon^2 + C\epsilon s_\epsilon(x)\big) \leq 0,$$

if only $C \gg 1$. Similarly, defining: $\bar{w}_\epsilon = u - u_\epsilon + v_\epsilon$, it follows that:

$$\bar{w}_\epsilon(x) - \big(1 - \gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} \bar{w}_\epsilon(y)\,\mathrm{d}y \geq \mathcal{O}(\epsilon s_\epsilon(x)) - a_\epsilon\epsilon^2 + \big(a_\epsilon\epsilon^2 + C\epsilon s_\epsilon(x)\big) \geq 0.$$

The claimed result follows now by an application of Lemma 3.3 below, since:

$$\|u - u_\epsilon\|_{\mathcal{C}(\bar{\mathcal{D}})} \leq \|v_\epsilon\|_{\mathcal{C}(\bar{\mathcal{D}})} \to 0 \quad \text{ as } \epsilon \to 0,$$

in view of $w_\epsilon \leq 0$ and $\bar{w}_\epsilon \geq 0$. ■

Lemma 3.3. *If $u \in \mathcal{C}(\bar{\mathcal{D}})$ satisfies: $u(x) - \big(1 - \gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\,\mathrm{d}y \leq 0$ for all $x \in \bar{\mathcal{D}}$, then there must be: $u \leq 0$ in $\bar{\mathcal{D}}$.*

Proof. Let $x_0 \in \bar{\mathcal{D}}$ be such that $u(x_0) = \max_{\bar{\mathcal{D}}} u$ and that, by contradiction: $u(x_0) > 0$. Then:

$$u(x_0) \leq \big(1 - \gamma s_\epsilon(x_0)\big)\fint_{B_\epsilon(x_0)\cap\mathcal{D}} u(y)\,\mathrm{d}y \leq \big(1 - \gamma s_\epsilon(x_0)\big)u(x_0) \leq u(x_0),$$

implying that $s_\epsilon(x_0) = 0$ and that $u = u(x_0)$ is constant on $B_\epsilon(x_0)$. Iterating this argument, we produce another maximizer $\bar{x}_0 \in \mathcal{D}$ such that $u(\bar{x}_0) = \max_{\bar{\mathcal{D}}} u$ and $\mathrm{dist}(\bar{x}_0, \partial\mathcal{D}) < \epsilon$, contradicting $s_\epsilon(\bar{x}_0) = 0$. ■

4. Convergence to viscosity solutions

The purpose of this section is to show that any uniform limit of a sequence in $\{u_\epsilon\}_{\epsilon\to 0}$, must be a viscosity solution to the Robin problem (RL), provided that $f \in \mathcal{C}(\bar{\mathcal{D}})$. According to [9, Definition 7.4], we have the following definition:

Definition 4.1. *Let $\mathcal{D} \subset \mathbb{R}^N$ be a $\mathcal{C}^1$-regular domain and assume that $\gamma > 0$ and $f \in \mathcal{C}(\bar{\mathcal{D}})$. We say that $u \in \mathcal{C}(\bar{\mathcal{D}})$ is a viscosity solution to (RL), provided that:*

(i) *(viscosity sub-solution property)*

if $x \in \mathcal{D}$ *and* $(p, X) \in J^{2,+}_{\bar{\mathcal{D}}} u(x)$ *then:* $-\,\mathrm{trace}X \leq f(x)$,

if $x \in \partial\mathcal{D}$ *and* $(p, X) \in J^{2,+}_{\bar{\mathcal{D}}} u(x)$ *then:* $-\,\mathrm{trace}X \leq f(x)$ *or* $\langle p, \vec{n}(x)\rangle + \gamma u(x) \leq 0$.

(ii) *(viscosity super-solution property)*

if $x \in \mathcal{D}$ *and* $(p,X) \in J^{2,-}_{\bar{\mathcal{D}}} u(x)$ *then:* $-\,traceX \geq f(x)$,

if $x \in \partial\mathcal{D}$ *and* $(p,X) \in J^{2,-}_{\bar{\mathcal{D}}} u(x)$ *then:* $-\,traceX \geq f(x)$ *or* $\langle p, \vec{n}(x)\rangle + \gamma u(x) \geq 0$.

The set of second order super-jets $J^{2,+}_{\bar{\mathcal{D}}} u(x)$ *consists of couples* $(p,X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}_{\mathrm{sym}}$, *satisfying:*

$$u(y) \leq u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle X : (y-x)^{\otimes 2}\rangle + o(|y-x|^2) \quad \text{as } \bar{\mathcal{D}} \ni y \to x. \tag{4.1}$$

Analogously, the set of second order sub-jets is defined by: $J^{2,-}_{\bar{\mathcal{D}}} u(x) = -J^{2,+}_{\bar{\mathcal{D}}}(-u)(x)$.

The following is the main statement of this section:

Theorem 4.2. *Assume* (BH) *and let* $f \in \mathcal{C}(\bar{\mathcal{D}})$. *If some sequence of solutions* $\{u_\epsilon\}_{\epsilon\to 0}$ *to* $(\mathrm{RMV})_\epsilon$ *converges uniformly on* $\bar{\mathcal{D}}$, *then the limit* $u \in \mathcal{C}(\bar{\mathcal{D}})$ *is a viscosity solution to* (RL).

Proof. **1.** Fix $x \in \bar{\mathcal{D}}$ and $(p,X) \in J^{2,+}_{\bar{\mathcal{D}}} u(x)$. For each $c>0$ consider the quadratic test function:

$$\phi_c(y) = u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle X : (y-x)^{\otimes 2}\rangle + \frac{c}{2}|y-x|^2.$$

Then: $u(x) = \phi_c(x)$, and $u < \phi_c$ in some neighbourhood of x in $\bar{\mathcal{D}}$ of the form: $(\bar{B}_{\epsilon_c}(x)\setminus\{x\}) \cap \bar{\mathcal{D}}$. Define the following sequence of positive numbers, decreasing to 0, as $j\to\infty$:

$$a_j = \min_{\big(\bar{B}_{\epsilon_c}(x)\setminus B_{1/j}(x)\big)\cap\bar{\mathcal{D}}} (\phi_c - u), \qquad \text{for all } j = 1,2\ldots$$

Let $\{\epsilon_j\}_{j\to\infty}$ be another sequence decreasing to 0, such that: $\|u_{\epsilon_j} - u\|_{\mathcal{C}(\bar{\mathcal{D}})} < \frac{1}{2}a_j$. Further, let:

$$x_j \in \bar{B}_{\epsilon_c}(x)\cap\bar{\mathcal{D}} \quad \text{satisfy:} \quad (\phi_c - u_{\epsilon_j})(x_j) = \min_{\bar{B}_{\epsilon_c}(x)\cap\bar{\mathcal{D}}} (\phi_c - u_{\epsilon_j}).$$

We immediately obtain that $x_j \in \bar{B}_{1/j}(x)\cap\bar{\mathcal{D}}$, so that $x_j \to x$ as $j\to\infty$, because:

$$\begin{aligned}(\phi_c - u_{\epsilon_j})(y) &> (\phi_c - u)(y) - \frac{1}{2}a_j \geq \frac{1}{2}a_j \\ &> u(x) - u_{\epsilon_j}(x) = (\phi_c - u_{\epsilon_j})(x) \quad \text{for all } y \in \big(\bar{B}_{\epsilon_c}(x)\setminus B_{1/j}(x)\big)\cap\bar{\mathcal{D}}.\end{aligned}$$

We now write $(\mathrm{RMV})_\epsilon$ as:

$$\begin{aligned} f(x_j) &= \frac{2(N+2)}{\epsilon_j^2}\Big(u_{\epsilon_j}(x_j) - \big(1-\gamma s_{\epsilon_j}(x_j)\big) \fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} u_{\epsilon_j}(y)\,\mathrm{d}y\Big) \\ &= \frac{2(N+2)}{\epsilon_j^2}\Big(\big(u_{\epsilon_j}(x_j) - \phi_c(x_j)\big) - \big(1-\gamma s_{\epsilon_j}(x_j)\big) \fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} u_{\epsilon_j} - \phi_c\,\mathrm{d}y \\ &\qquad\qquad + \phi_c(x_j) - \big(1-\gamma s_{\epsilon_j}(x_j)\big) \fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} \phi_c(y)\,\mathrm{d}y\Big) \\ &= \frac{2(N+2)}{\epsilon_j^2}\big(I_j + II_j\big). \end{aligned} \tag{4.2}$$

By the definition of x_j as a local minimizer of $\phi_c - u_{\epsilon_j}$, it follows that:

$$I_j \geq \big(u_{\epsilon_j}(x_j) - \phi_c(x_j)\big)\gamma s_{\epsilon_j}(x_j). \tag{4.3}$$

The bound on II_j will be achieved separately in the interior and the boundary cases.

2. If $x \in \mathcal{D}$, then $s_{\epsilon_j}(x_j) = 0$ for all $j \gg 1$, implying that $I_j \geq 0$ by (4.3). Consequently:

$$II_j = \phi_c(x_j) - \fint_{B_{\epsilon_j}(x_j)} \phi_c(y)\;\mathrm{d}y = -\frac{\epsilon_j^2}{2(N+2)}\Delta\phi_c(x) = -\frac{\epsilon_j^2}{2(N+2)}\big(\mathrm{trace}X + cN\big).$$

Thus (4.2) becomes: $f(x_j) \geq -\mathrm{trace}X - cN$. Passing to the limit with $j \to \infty$ and $c \to 0$, we obtain the condition requested in Definition 4.1 (i):

$$-\mathrm{trace}X \leq f(x).$$

The same reasoning shows that $-\mathrm{trace}X \geq f(x)$ for all $(p, X) \in J^{2,-}_{\bar{\mathcal{D}}}u(x)$.

3. Assume that $x \in \partial\mathcal{D}$. If for some subsequence we have: $\epsilon_j \leq \mathrm{dist}(x_j, \partial\mathcal{D})$, then: $-\mathrm{trace}X \leq f(x)$ by the same reasoning as in step 2 above. Therefore, it suffices to assume:

$$\mathrm{dist}(x_j, \partial\mathcal{D}) = \epsilon_j d_j \quad \text{ where } \; d_j = d_{\epsilon_j}(x_j) \in [0, 1) \quad \text{ for all } \; j = 1, 2 \ldots$$

Call $s_j = s_{\epsilon_j}(x_j)$ and rewrite (4.2) as follows:

$$\begin{aligned} f(x_j) = \frac{2(N+2)}{\epsilon_j^2}\Big(&\big(u_{\epsilon_j}(x_j) - \phi_c(x_j)\big)\gamma s_j + \phi_c(x_j)\gamma s_j \\ &- (1-\gamma s_j)\fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} \phi_c(y) - \phi_c(x_j)\;\mathrm{d}y\Big). \end{aligned} \tag{4.4}$$

We will assume the following condition:

$$\langle p, \vec{n}(x)\rangle + \gamma u(x) > 0, \tag{4.5}$$

which, in particular, implies that the same quantity remains bounded away from 0, if $\vec{n}(x)$ is replaced by $\vec{n}(\pi_{\partial\mathcal{D}}x_j)$ and $u(x)$ by $\phi_c(x_j)$. Consequently, for some $a > 0$ there holds:

$$\phi_c(x_j) \geq 2a - \frac{1}{\gamma}\big\langle p, \vec{n}(\pi_{\partial\mathcal{D}}x_j)\big\rangle \quad \text{ for all } \; j = 1, 2 \ldots$$

and (4.4) becomes, in view of $\nabla\phi_c(x_j) = p + (X + cId_N)(x_j - x)$ and $\nabla^2\phi_c(x_j) = X + cId_N$:

$$\begin{aligned} f(x_j) \geq \frac{2(N+2)}{\epsilon_j^2}\Big(&a\gamma s_j - s_j\big\langle p, \vec{n}(\pi_{\partial\mathcal{D}}x_j)\big\rangle - \big\langle p, \fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} y - x_j\;\mathrm{d}y\big\rangle \\ &- \big\langle (X + cId_N)(x_j - x), \fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} y - x_j\;\mathrm{d}y\big\rangle \\ &- \frac{1}{2}\big\langle X + cId_N : \fint_{B_{\epsilon_j}(x_j)\cap\mathcal{D}} (y - x_j)^{\otimes 2}\;\mathrm{d}y\big\rangle\Big). \end{aligned}$$

We now use (2.3) and (2.5) to get, for all $j \gg 1$:

$$\begin{aligned} f(x_j) &\geq \frac{2(N+2)}{\epsilon_j^2}\Big(a\gamma s_j - \frac{\epsilon_j^2}{2(N+2)}\big\langle X + cId_N : Id_N\big\rangle + \mathcal{O}(|x_j - x|s_j) + \mathcal{O}(\epsilon_j s_j)\Big) \\ &\geq -\mathrm{trace}X - cN. \end{aligned}$$

Passing to the limit with $j \to \infty$ and with $c \to 0$, we obtain: $-\mathrm{trace}X \leq f(x)$, which is precisely the condition requested in Definition 4.1 (i), now pertaining to the case $x \in \partial\mathcal{D}$, in presence of (4.5). A similar reasoning yields that u is also a viscosity super-solution. ∎

In fact, viscosity solutions to (RL) are unique under the uniform outer supporting sphere assumption. This statement follows from analysis in [9, Theorem 7.5] but since the linear operator $-\Delta$ does not satisfy the u-coercivity assumption (7.14) in there, we will sketch the related proof of comparison principle in Lemma 9.2 in the appendix section 9. In section 7 we will show that the entire sequence $\{u_\epsilon\}_{\epsilon\to 0}$ converges to the unique $W^{2,p}$ solution to (RL). Thus, we independently obtain that viscosity solutions exist, are unique and coincide with the weak solutions. We remark that another proof of existence of viscosity solutions can be obtained by Perron's method in view of the comparison principle [9, 13]. Finally, we anticipate that in [17] we show the asymptotic Hölder equicontinuity of t$\{u_\epsilon\}_{\epsilon\to 0}$ generated by any Borel $f\in L^\infty(\bar{\mathcal{D}})$. Combined with a refinement of the present arguments, this yields the assertion of Theorem 1.1 (ii) and thus another independent proof of existence and uniqueness of solutions to (RL), in the more general case.

5. The first probabilistic interpretation of u_ϵ

In this section, we develop the basic probability setting related to the equation $(\mathrm{RMV})_\epsilon$.

1. Consider the probability space $(B_1^N, \mathcal{B}, \frac{1}{|B_1^N|}\mathcal{L}_N)$ equipped with the standard Borel σ-algebra and the normalised Lebesgue measure, and define $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ as the countable product of B_1^N augmented by the unit interval (likewise equipped with Borel σ-algebra and Lebesgue measure):

$$\Omega_1 = (B_1^N)^{\mathbb{N}}\times(0,1) = \big\{(w,b);\ w=\{w^j\}_{j=1}^\infty,\ w^j\in B_1^N\quad \text{for all } j\in\mathbb{N}\quad\text{and } b\in(0,1)\big\}.$$

Further, the countable product of $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ is denoted by $(\Omega,\mathcal{F},\mathbb{P})$, where:

$$\Omega = (\Omega_1)^{\mathbb{N}} = \big\{\omega = \{(w_i,b_i)\}_{i=1}^\infty;\ w_i=\{w_i^j\}_{j=1}^\infty,\ w_i^j\in B_1^N,\ b_i\in(0,1)\quad\text{for all } i,j\in\mathbb{N}\big\}.$$

For each $n\in\mathbb{N}$, the probability space $(\Omega_n,\mathcal{F}_n,\mathbb{P}_n)$ is the product of n copies of $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ and the σ-algebra $\mathcal{F}_n$ is identified with the sub-σ-algebra of $\mathcal{F}$, consisting of sets $A\times\prod_{i=n+1}^\infty\Omega_1$ for all $A\in\mathcal{F}_n$. Then $\{\mathcal{F}_n\}_{n=0}^\infty$ where $\mathcal{F}_0=\{\emptyset,\Omega\}$, is a filtration of $\mathcal{F}$.

2. Given $\epsilon\ll 1$, define the sequence of measurable functions $\big\{k_i^\epsilon:\Omega\times\bar{\mathcal{D}}\to\mathbb{N}\cup\{+\infty\}\big\}_{i=1}^\infty$ by:

$$k_i^\epsilon(\omega,x) = \min\big\{k\geq 1;\ x+\epsilon w_i^k\in B_\epsilon(x)\cap\mathcal{D}\big\}\qquad\text{for all }\ \omega\in\Omega,\ x\in\bar{\mathcal{D}}.$$

Since each k_i^ϵ is $\mathbb{P}$-a.s. finite, we further construct the sequence of vector-valued random variables $\{w_i^{\epsilon,x}:\Omega\to B_1^N\}_{i=1}^\infty$, corresponding to $\epsilon\ll 1$ and $x\in\bar{\mathcal{D}}$, by:

$$w_i^{\epsilon,x}(\omega) = w_i^{k_i^\epsilon(\omega,x)}\qquad\text{for }\mathbb{P}\text{-a.e. }\ \omega\in\Omega.$$

The procedure of generating $w_i^{k_i}$ is well known under the name of rejection sampling and it has the following measure preservation property: for every ϵ, x as above, for every Borel set $F\subset B_\epsilon(x)\cap\mathcal{D}$:

$$\begin{split}\mathbb{P}\big(x+\epsilon w_i^{\epsilon,x}\in F\big) &= \sum_{k=1}^\infty\mathbb{P}_1\big(\{x+\epsilon w_i^k\in F\}\cap\{k=k_i^\epsilon\}\big)\\ &= \frac{|F|}{|B_\epsilon(x)|}\cdot\sum_{k=1}^\infty\Big(1-\frac{|B_\epsilon(x)\cap\mathcal{D}|}{|B_\epsilon(x)|}\Big)^{k-1} = \frac{|F|}{|B_\epsilon(x)\cap\mathcal{D}|}.\end{split}\tag{5.1}$$

For $\epsilon\ll 1$, $x_0\in\bar{\mathcal{D}}$, we recursively define the sequence of random variables $\big\{X_n^{\epsilon,x_0}:\Omega\to\bar{\mathcal{D}}\big\}_{n=0}^\infty$:

$$X_0^{\epsilon,x_0}\equiv x_0,\qquad X_n^{\epsilon,x_0}\big(w_1,\dots,w_n\big) = X_{n-1}^{\epsilon,x_0}\big(w_1,\dots,w_{n-1}\big)+\epsilon w_n^{\epsilon,X_{n-1}^{\epsilon,x_0}(w_1,\dots,w_{n-1})}.\tag{5.2}$$

Clearly, the function X_n^{ϵ,x_0} is $\mathcal{F}_n$-measurable and takes values in $\mathcal{D}$, for $n\geq 1$. Given $\gamma>0$, define further the $\mathcal{F}$-measurable $\tau^{\epsilon,x_0}:\Omega\to\mathbb{N}\cup\{+\infty\}$ by:

$$\tau^{\epsilon,x_0}(\omega) = \min\big\{n\geq 1;\ b_n<\gamma s_\epsilon(X_{n-1}^{\epsilon,x_0})\big\}.$$

We observe that each $X_n^{\epsilon,x_0}(\omega)$ and τ^{ϵ,x_0} are jointly measurable in ω and x_0, by the same property of k_i^ϵ. When no ambiguity arises, we will simply write $X_n^{x_0}$ and τ^{x_0}, to simplify the notation.

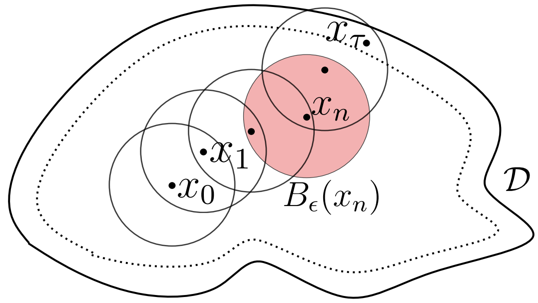

FIGURE 5.1. Positions of the process defined in (5.2).

3. The following elementary argument proves that τ^{x_0} is finite $\mathbb{P}$-a.e., making it a stopping time. It is easy to notice that there exists $\bar n \gg 1$ and $\delta > 0$ such that for every $x_0 \in \bar{\mathcal{D}}$ there is some $n \leq \bar n$, satisfying:

$$\mathbb{P}_{n-1}\Big((w_1,\ldots,w_{n-1});\ X_{n-1}^{x_0}(w_1,\ldots,w_{n-1}) \in \partial\mathcal{D} + B_{\epsilon/2}(0)\Big) \geq \delta.$$

Indeed, given $x_0 \in \bar{\mathcal{D}}$, one may choose $\bar x \in \partial\mathcal{D}$ such that $\mathrm{dist}(x_0,\partial\mathcal{D}) = |x_0 - \bar x|$ and keep advancing the position $X_i^{x_0}$ in the direction of $\bar x$, by random increments w^1 within a small sector of positive measure, located in $\frac{1}{2}B_1^N$ close to its boundary, until $X_i^{x_0} \in \partial\mathcal{D}+B_{\epsilon/2}(0)$. Further, since $1 \geq \gamma s_\epsilon(x) \geq C\epsilon$ for all $x \in (\partial\mathcal{D} + B_{\epsilon/2}(0)) \cap \mathcal{D}$ and $\epsilon \ll 1$, we get: $\mathbb{P}(\tau^{x_0} \leq \bar n) \geq (C\epsilon)^{\bar n}\delta = \bar\delta > 0$. By induction, if follows that: $\mathbb{P}\big(\tau^{x_0} > k\bar n\big) \leq (1-\bar\delta)^k$ for all $k \geq 0$, and so $\mathbb{P}\big(\tau^{x_0} = \infty\big) = 0$, completing the argument. It also follows that:

$$\mathbb{E}\big[\tau^{x_0}\big] = \sum_{i=0}^{\infty} \mathbb{P}\big(\tau^{x_0} > i\big) \leq \bar n \cdot \sum_{k=0}^{\infty} \mathbb{P}\big(\tau^{x_0} > k\bar n\big) \leq \frac{\bar n}{\bar\delta} = C_\epsilon. \tag{5.3}$$

In fact, the above constant C_ϵ is of the order $\frac{C}{\epsilon^2}$, as noted in Corollary 5.2.

4. For each $\epsilon \ll 1, x_0 \in \bar{\mathcal{D}}$ we define the random variable $F^{\epsilon,x_0}(\omega) = \sum_{i=0}^{\tau^{x_0}-1} \big(f \circ X_i^{x_0}\big)(\omega)$ that is $\mathbb{P}$-integrable in view of (5.3), and set:

$$u^\epsilon(x_0) = \frac{\epsilon^2}{2(N+2)}\mathbb{E}\big[F^{\epsilon,x_0}\big] = \int_\Omega \sum_{i=0}^{\tau^{x_0}-1} \frac{\epsilon^2}{2(N+2)}\big(f \circ X_i^{x_0}\big)(\omega)\ \mathrm{d}\mathbb{P}(\omega). \tag{5.4}$$

Since the function $\Omega\times\bar{\mathcal{D}} \ni (\omega, x_0) \mapsto F^{x_0}(\omega)$ is measurable, in view of the same joint measurability of $X_n^{x_0}$ and τ^{x_0}, it follows that u^ϵ is a Borel function of x_0. The following is the main observation:

Lemma 5.1. *Assume (BH). For each $\epsilon \ll 1$, the function u^ϵ in (5.4) coincides with the unique bounded, Borel solution to $(\mathrm{RMV})_\epsilon$:*

$$u^\epsilon = u_\epsilon.$$

Proof. Boundedness of u^ϵ results from (5.3), because: $|u^\epsilon(x_0)| \leq \|f\|_{L^\infty(\mathcal{D})}\epsilon^2 C_\epsilon$ for every $x_0 \in \bar{\mathcal{D}}$. To check that u^ϵ satisfies $(\mathrm{RMV})_\epsilon$, observe that:

$$F^{x_0}\big((w_1,b_1),(w_2,b_2),\ldots\big) = f(x_0) + \mathbb{1}_{b_1\geq\gamma s_\epsilon(x_0)} \cdot \sum_{i=0}^{\tau^{X_1^{x_0}(w_1)}-1} f\big(X_i^{X_1^{x_0}(w_1)}((w_2,b_2),(w_3,b_3),\ldots\big).$$

Consequently:

$$\begin{aligned} u^\epsilon(x_0) &= \frac{\epsilon^2}{2(N+2)}\mathbb{E}\big[F^{x_0}\big] \\ &= \frac{\epsilon^2}{2(N+2)}\int_{\Omega_1}\int_{\prod_{i=2}^\infty \Omega_1} F^{x_0}\big((w_1,b_1),(w_2,b_2),\dots\big)\ \mathrm{d}\big(\prod_{i=2}^\infty \mathbb{P}_1(w_i,b_i)\big)\ \mathrm{d}\mathbb{P}_1(w_1,b_1) \\ &= \frac{\epsilon^2}{2(N+2)}f(x_0) + \int_{\Omega_1}\big(1-\gamma s_\epsilon(x_0)\big)\int_{\prod_{i=2}^\infty \Omega_1}\sum_{i=0}^{\tau^{X_1}-1}\frac{\epsilon^2}{2(N+2)}f(X_i^{X_1})\ \mathrm{d}\big(\prod_{i=2}^\infty \mathbb{P}_1\big)\ \mathrm{d}\mathbb{P}_1(w_1,b_1) \\ &= \frac{\epsilon^2}{2(N+2)}f(x_0) + \big(1-\gamma s_\epsilon(x_0)\big)\int_{\Omega_1} u^\epsilon\big(X_1^{x_0}(w_1)\big)\ \mathrm{d}\mathbb{P}_1(w_1). \end{aligned}$$

Changing the variable in the last integral and recalling the measure-preserving property (5.1) in:

$$\begin{aligned} u^\epsilon(x_0) &= \frac{\epsilon^2}{2(N+2)}f(x_0) + \big(1-\gamma s_\epsilon(x_0)\big)\int_{\Omega_1} u^\epsilon\big(x_0+\epsilon w_1^{k_1^\epsilon(w_1,x_0)}\big)\ \mathrm{d}\mathbb{P}_1(w_1) \\ &= \frac{\epsilon^2}{2(N+2)}f(x_0) + \big(1-\gamma s_\epsilon(x_0)\big)\fint_{B_\epsilon(x_0)\cap\mathcal{D}} u^\epsilon(y)\ \mathrm{d}y, \end{aligned}$$

yields $u^\epsilon = u_\epsilon$ by the uniqueness of solutions to (RMV)$_\epsilon$. ■

We finally deduce the following refinement of (5.3). In section 8 we then prove a precise lower bound on $\mathbb{E}\big[\tau^{\epsilon,x_0}\big]$ in terms of $\frac{1}{\epsilon^2}$ and the parameters γ and r in (BH).

Corollary 5.2. *There exists a constant C, depending on $\mathcal{D}$ and γ but not on ϵ, such that:*

$$\mathbb{E}\big[\tau^{\epsilon,x_0}\big] \leq \frac{C}{\epsilon^2} \qquad \textit{for all } \ x_0\in\bar{\mathcal{D}} \quad \textit{and all } \ \epsilon\ll 1.$$

Proof. We have:

$$\mathbb{E}\big[\tau^{\epsilon,x_0}\big] - 1 = \frac{2(N+2)}{\epsilon^2}u_\epsilon^f(x_0) \leq \frac{C}{\epsilon^2},$$

where we used Lemma 5.1 and Theorem 2.5 applied to the constant function $f=1$. ■

Remark 5.3. Another way of defining the process in (5.2) is based on the quantile regression procedure, as we now describe. We first redefine the probability spaces, by setting $\Omega_1 = B_1^N\times(0,1)$ and taking $\mathcal{F}_1$ to be the Borel σ-algebra and $\mathbb{P}_1$ the normalised Lebesgue measure on Ω_1. The probability space $(\Omega,\mathcal{F},\mathbb{P})$ is given as the countable product of $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$, where:

$$\Omega = (\Omega_1)^{\mathbb{N}} = \big\{\omega = \{(w_i,b_i)\}_{i=1}^\infty;\ w_i\in B_1^N,\ b_i\in(0,1) \quad \text{for all } i\in\mathbb{N}\big\}.$$

For each $n\in\mathbb{N}$, we have the corresponding $(\Omega_n,\mathcal{F}_n,\mathbb{P}_n)$ as before. For $x_0\in\bar{\mathcal{D}}$, $\epsilon\ll1$, we now inductively define the sequence of random variables: $\big\{X_n^{\epsilon,x_0}:\Omega\to\bar{\mathcal{D}}\big\}_{n=0}^\infty$:

$$X_0^{\epsilon,x_0}\equiv x_0, \qquad X_n^{\epsilon,x_0}\big(w_1,\dots,w_n\big) = X_{n-1}^{\epsilon,x_0}\big(w_1,\dots,w_{n-1}\big) + T_\epsilon\big(X_{n-1}^{\epsilon,x_0},\epsilon w_n\big), \tag{5.5}$$

where the transformation T_ϵ is given by the classical Knothe-Rosenblatt rearrangement [27, 14], whose construction we sketch below. Each X_n^{ϵ,x_0} in (5.5) is $\mathcal{F}_n$-measurable and it takes values in $\bar{\mathcal{D}}$. Given $\gamma>0$, let:

$$\tau^{\epsilon,x_0}(\omega) = \min\big\{n\geq1;\ b_n<\gamma s_\epsilon(X_{n-1}^{\epsilon,x_0})\big\} \qquad \text{for all } \ \omega\in\Omega,$$

and observe, as in (5.3) that it is a stopping time. With the same definition of u^ϵ as in (5.4), the statement in Lemma 5.1 is valid as before.

Lemma 5.4. *Let $\mathcal{D} \subset \mathbb{R}^N$ be an open, bounded, $\mathcal{C}^1$-regular domain. For all $\epsilon \ll 1$ there exists a continuous transformation $T_\epsilon : \bar{\mathcal{D}} \times B_\epsilon(0) \to \mathbb{R}^N$ such that for every $x \in \bar{\mathcal{D}}$, the map $T_\epsilon(x, \cdot) : B_\epsilon(0) \to B_\epsilon(x) \cap \mathcal{D}$ is a homeomorphism that is normalised-volume preserving, i.e.:*

$$\frac{|T_\epsilon(x,F)|}{|B_\epsilon(x) \cap \mathcal{D}|} = \frac{|F|}{|B_\epsilon(0)|} \qquad \textit{for every Borel set } \; F \subset B_\epsilon(0).$$

Proof. Fix $x_0 \in \bar{\mathcal{D}}$. For $\mathrm{dist}(x_0, \partial\mathcal{D}) \geq \epsilon$, we set $T_\epsilon(x_0, \cdot) = id_N$. When $\mathrm{dist}(x_0, \partial\mathcal{D}) < \epsilon \ll 1$ then $\partial\mathcal{D} \cap B_\epsilon(x_0)$ is a $\mathcal{C}^1$ graph over the hyperplane perpendicular to $\vec{n}$, and by rescaling and rotating, we reduce to the case:

$$x_0 = 0, \quad \epsilon = 1, \quad \vec{n} = e_N, \quad B_\epsilon(0) \cap \mathcal{D} = B_1(0) \cap \{x_N < \alpha(x_1, \ldots, x_{N-1})\}.$$

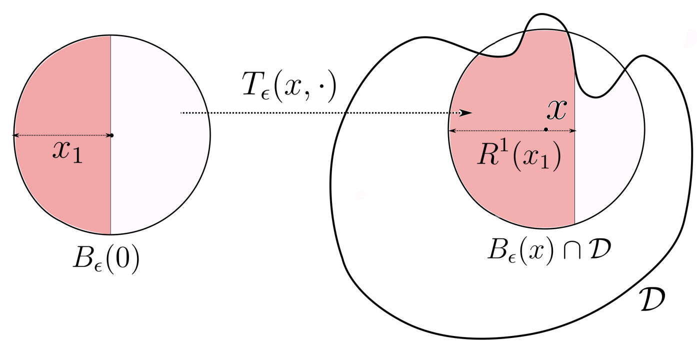

FIGURE 5.2. The Knothe-Rosenblatt rearrangement in Lemma 5.4.

The Knothe-Rosenblatt rearrangement map is best described by denoting:

$$\bar{\alpha} = \frac{\mathbb{1}_{B_1^N \cap \{x_N < \alpha(x_1,\ldots,x_{N-1})\}}}{|B_1^N \cap \{x_N < \alpha(x_1, \ldots, x_{N-1})\}|}, \qquad \bar{\beta} = \frac{\mathbb{1}_{B_1^N}}{|B_1^N|}$$

and seeking the measure-preserving diffeomorphism $R = T_1(x_0, \cdot)$:

$$R : \big(B_1^N, \bar{\beta}\, \mathrm{d}x\big) \to \big(B_1^N \cap \{x_N < \alpha\}, \bar{\alpha}\, \mathrm{d}x\big),$$

of the form: $R(x_1, \ldots, x_N) = \big(R^1(x_1), R^2(x_1, x_2), \ldots, R^N(x_1, \ldots, x_N)\big)$. The continuous and strictly increasing components $R^i(x_1, \ldots, x_{i-1}, \cdot)$ are recursively defined by implicit formulas:

$$\int_{-\infty}^{R^1(x_1)} \int_{\mathbb{R}^{N-1}} \bar{\alpha}\big(y_1, y_2, \ldots, y_N\big)\, \mathrm{d}(y_2, \ldots, y_N)\, \mathrm{d}y_1 = \int_{-\infty}^{x_1} \int_{\mathbb{R}^{N-1}} \bar{\beta}\big(y_1, y_2, \ldots, y_N\big)\, \mathrm{d}(y_2, \ldots, y_N)\, \mathrm{d}y_1$$

and, for all $x_1 \in \mathbb{R}$:

$$\begin{aligned} &\int_{-\infty}^{R^2(x_1,x_2)} \int_{\mathbb{R}^{N-2}} \bar{\alpha}\big(R^1(x_1), y_2, \ldots, y_N\big)\, \mathrm{d}(y_3, \ldots, y_N)\, \mathrm{d}y_2 \\ &\quad = \frac{\int_{\mathbb{R}^{N-1}} \bar{\alpha}(R^1(x_1), y_2, \ldots, y_N)\, \mathrm{d}(y_2, \ldots, y_N)}{\int_{\mathbb{R}^{N-1}} \bar{\beta}\big(x_1, y_2, \ldots, y_N\big)\, \mathrm{d}(y_2, \ldots, y_N)} \cdot \int_{-\infty}^{x_2} \int_{\mathbb{R}^{N-2}} \bar{\beta}\big(x_1, y_2, \ldots, y_N\big)\, \mathrm{d}(y_3, \ldots, y_N)\, \mathrm{d}y_2, \end{aligned}$$

eventually followed by the identity:

$$\begin{aligned} &\int_{-\infty}^{R^N(x_1,\ldots,x_N)} \int_{\mathbb{R}^{N-2}} \bar{\alpha}\big(R^1(x_1), R^2(x_1, x_2), \ldots, R^{N-1}(x_1, \ldots, x_{N-1}, y_N\big)\, \mathrm{d}y_N \\ &\quad = \frac{\int_{\mathbb{R}} \bar{\alpha}\big(R^1(x_1), R^2(x_1, x_2), \ldots, R^{N-1}(x_1, \ldots, x_{N-1}), y_N\big)\, \mathrm{d}y_N}{\int_{\mathbb{R}} \bar{\beta}\big(x_1, \ldots, x_{N-1}, y_N\big)\, \mathrm{d}y_N} \cdot \int_{-\infty}^{x_N} \bar{\beta}\big(x_1, \ldots, x_{N-1}, y_N\big)\, \mathrm{d}y_N, \end{aligned}$$

whose validity is requested for all $(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. The fact that R is measure preserving is then classical and may be checked directly. Continuity of R and of its inverse R^{-1} is implied by the property that sections of both B_1^N and $B_1^N \cap \{x_N < \alpha\}$, obtained by fixing $(x_1, \ldots, x_i)$, are connected sets, for any $i \leq N-1$. Finally, $R \equiv id_N$ when $\alpha = 1$ and the construction is continuous with respect to $\vec{n}$, justifying continuity of T_ϵ in x_0. ■

6. The second probabilistic interpretation of u_ϵ

We now propose another formula for solutions of $(\mathrm{RMV})_\epsilon$, parallel to (5.4). For each $\epsilon \ll 1$, $x_0 \in \bar{\mathcal{D}}$, define the random variables $\big\{\Lambda_n^{\epsilon,x_0} : \Omega \to \mathbb{R}\big\}_{n=0}^\infty$ along the process $\{X_n^{\epsilon,x_0}\}_{n=0}^\infty$ in (5.2):

$$\Lambda_n^{\epsilon,x_0}(\omega) = \prod_{j=1}^n \big(1 - \gamma(s_\epsilon \circ X_{j-1}^{\epsilon,x_0})(\omega)\big), \tag{6.1}$$

For $n = 0$ we adopt the convention that $\Lambda_0 = 1$. When no ambiguity arises we write $\Lambda_n^{x_0}$ or Λ_n, and note that each $\Lambda_n^{x_0}(\omega)$ is jointly measurable in ω and x_0. We now set:

$$\begin{aligned} \bar{u}^\epsilon(x_0) &= \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{i=0}^\infty (f\circ X_i^{x_0})\Lambda_i^{x_0}\Big] \\ &= \int_\Omega \sum_{i=0}^\infty \frac{\epsilon^2}{2(N+2)}\big(f\circ X_i^{x_0}\big)(\omega)\cdot \prod_{j=1}^i \big(1-\gamma(s_\epsilon\circ X_{j-1}^{\epsilon,x_0})(\omega)\big)\,\mathrm{d}\mathbb{P}(\omega). \end{aligned} \tag{6.2}$$

Lemma 6.1. *Assume* (BH). *For each $\epsilon \ll 1$, the function $\bar{u}^\epsilon$ is well defined a.e. in $\bar{\mathcal{D}}$. After a possible adjustment on a negligible set, it coincides with unique bounded Borel solution to $(\mathrm{RMV})_\epsilon$:*

$$\bar{u}^\epsilon = u_\epsilon.$$

Proof. **1.** Denote by v_ϵ the solution to $(\mathrm{RMV})_\epsilon$ with $f \equiv 1$. We first argue that the following sequence of random variables $\{M_n\}_{n=0}^\infty$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$:

$$M_n = (v_\epsilon \circ X_n)\Lambda_n + \frac{\epsilon^2}{2(N+2)}\sum_{i=0}^{n-1}\Lambda_i,$$

where we adopt the convention that $M_0 = v_\epsilon(x_0)$. Indeed, $(\mathrm{RMV})_\epsilon$ yields:

$$\begin{aligned} \mathbb{E}\big(M_{n+1} - M_n \mid \mathcal{F}_n\big) &= \Big(\fint_{B_\epsilon(X_n)\cap\mathcal{D}} v_\epsilon(y)\,\mathrm{d}y\Big)\cdot\Lambda_{n+1} - v_\epsilon(X_n)\cdot\Lambda_n + \frac{\epsilon^2}{2(N+2)}\Lambda_n \\ &= \Big(\big(1-\gamma s_\epsilon(X_n)\big)\fint_{B_\epsilon(X_n)\cap\mathcal{D}} v_\epsilon(y)\,\mathrm{d}y - v_\epsilon(X_n) + \frac{\epsilon^2}{2(N+2)}\Big)\Lambda_n \\ &= 0 \qquad \mathbb{P}-\text{a.s. in } \Omega. \end{aligned}$$

Using equiboundedness of $\{v_\epsilon\}_{\epsilon\to 0}$ in Theorem 2.5 we get:

$$\frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{i=0}^n \Lambda_i\Big] \leq \mathbb{E}\big[M_{n+1}\big] = \mathbb{E}\big[M_0\big] = v_\epsilon(x_0) \leq C. \tag{6.3}$$

The above uniform (independent of x_0, ϵ and n) bound suffices to conclude that the function $\Omega\times\bar{\mathcal{D}} \ni (\omega, x_0) \mapsto \frac{\epsilon^2}{2(N+2)}\sum_{i=0}^\infty (f\circ X_i^{x_0})(\omega)\Lambda_i^{x_0}(\omega)$ is jointly integrable. Consequently, $\bar{u}^\epsilon$ is well defined for a.e. $x_0 \in \bar{\mathcal{D}}$, Borel regular and bounded, and moreover:

$$\mathbb{E}\Big[\sum_{i=0}^\infty (f\circ X_i^{x_0})\Lambda_i^{x_0}\Big] = \sum_{i=0}^\infty \mathbb{E}\Big[(f\circ X_i^{x_0})\Lambda_i^{x_0}\Big] \tag{6.4}$$

2. Similarly as in Lemma 5.1, Fubini's theorem and change of variable in view of (5.1) yield:

$$\begin{aligned}\bar u^\epsilon(x_0) &= \frac{\epsilon^2}{2(N+2)}\bigg(f(x_0) + \big(1-\gamma s_\epsilon(x_0)\big)\int_{\Omega_1}\mathbb{E}\Big[\sum_{i=0}^\infty (f\circ X_i^{X_1})\cdot\prod_{j=1}^i\big(1-\gamma s_\epsilon(X_{j-1}^{X_1})\big)\Big]\,\mathrm{d}\mathbb{P}_1\bigg)\\ &= \frac{\epsilon^2}{2(N+2)}f(x_0) + \big(1-\gamma s_\epsilon(x_0)\big)\int_{\Omega_1}\bar u^\epsilon\big(X_1^{x_0}(w_1)\big)\,\mathrm{d}\mathbb{P}_1(w_1)\\ &= \frac{\epsilon^2}{2(N+2)}f(x_0) + \big(1-\gamma s_\epsilon(x_0)\big)\fint_{B_\epsilon(x_0)\cap\mathcal{D}}\bar u^\epsilon(y)\,\mathrm{d}y,\end{aligned}$$

for a.e. $x_0\in\bar{\mathcal{D}}$. By possibly redefining $\bar u^\epsilon$ on a Borel set of measure zero in $\bar{\mathcal{D}}$, we obtain the validity of $(\mathrm{RMV})_\epsilon$ for all $x_0\in\bar{\mathcal{D}}$, as claimed. ■

We conclude this section by estimating the expectation of the accumulated Dirichlet factors Λ_n, in the boundary layer where $d_\epsilon<1$. This property will be used in section 7, towards bounding the error of $\bar u^\epsilon$ from the first order Taylor expansion of the weak solution u to (RL).

Lemma 6.2. *Assume (BH). Then for each $\epsilon\ll 1$ and $x_0\in\bar{\mathcal{D}}$, the function $\sum_{i=0}^\infty \mathbb{1}_{\{d_\epsilon(X_k)<1\}}\Lambda_k^{x_0}$ is an integrable random variable and we have:*

$$\mathbb{E}\Big[\sum_{i=0}^\infty \mathbb{1}_{\{d_\epsilon(X_i^{x_0})<1\}}\Lambda_i^{x_0}\Big]\leq\frac{C}{\epsilon},$$

where C is a constant that is independent of ϵ and x_0, but may depend on f, $\mathcal{D}$ and γ.

Proof. Fix $\epsilon\ll 1$, $x_0\in\bar{\mathcal{D}}$. Given a constant $\lambda>0$, we consider the sequence of random variables:

$$M_n=\sum_{i=0}^{n-1}\Big(\lambda\epsilon\mathbb{1}_{\{d_\epsilon(X_i^{x_0})<1\}} - s_\epsilon\circ X_{i+1}^{x_0}\Big)\Lambda_i,$$

where we adopt the convention that $M_0=0$. We claim that $\{M_n\}_{n=0}^\infty$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$, provided that $\lambda=\lambda(\mathcal{D})$ is chosen appropriately. Indeed:

$$\begin{aligned}\mathbb{E}\big(M_{n+1}-M_n\mid\mathcal{F}_n\big) &= \mathbb{E}\Big(\big(\lambda\epsilon\mathbb{1}_{\{d_\epsilon(X_n)<1\}} - s_\epsilon(X_{n+1})\big)\Lambda_n\mid\mathcal{F}_n\Big)\\ &= \Big(\lambda\mathbb{1}_{\{d_\epsilon(X_n)<1\}} - \frac{1}{\epsilon}\mathbb{E}\big(s_\epsilon(X_{n+1})\mid\mathcal{F}_n\big)\Big)\epsilon\Lambda_n\leq 0 \qquad \mathbb{P}-\text{a.s. in }\Omega,\end{aligned}$$

where the last inequality follows by observing that on the event $\{\mathrm{dist}(X_n,\partial\mathcal{D})\geq\epsilon\}$ the quantity in parentheses is clearly nonpositive, whereas on the event $\{\mathrm{dist}(X_n,\partial\mathcal{D})<\epsilon\}$ it is nonpositive, upon choosing λ small so that:

$$\begin{aligned}\frac{1}{\epsilon}\mathbb{E}\big(s_\epsilon(X_{n+1})\mid\mathcal{F}_n\big) &= \frac{1}{\epsilon}\fint_{B_\epsilon(X_n)\cap\mathcal{D}} s_\epsilon(y)\,\mathrm{d}y\\ &\geq c\fint_{B_\epsilon(X_n)\cap\mathcal{D}}\big(1-d_\epsilon(y)^2\big)^{\frac{N+1}{2}}\,\mathrm{d}y\geq\lambda \qquad \mathbb{P}-\text{a.s. in }\Omega.\end{aligned}$$

The supermartingale property implies: $\mathbb{E}\big[M_{n+1}\big] \leq \mathbb{E}\big[M_0\big] = 0$ and thus we get:

$$\begin{aligned}\mathbb{E}\Big[\sum_{i=0}^{n} \epsilon \mathbb{1}_{\{d_\epsilon(X_i)<1\}}\Lambda_i^{x_0}\Big] &\leq \frac{1}{\lambda}\mathbb{E}\Big[\sum_{i=0}^{n} s_\epsilon(X_{i+1})\Lambda_i\Big] \leq \frac{2}{\lambda}\mathbb{E}\Big[\sum_{i=0}^{n} s_\epsilon(X_{i+1})\Lambda_{i+1}\Big] \\ &= \frac{2}{\lambda\gamma}\mathbb{E}\Big[\sum_{i=0}^{n} \big(1-(1-\gamma s_\epsilon(X_{i+1}))\big)\Lambda_{i+1}\Big] \\ &= \frac{2}{\lambda\gamma}\mathbb{E}\Big[\sum_{i=0}^{n}\Lambda_{i+1} - \sum_{i=0}^{n}\Lambda_{i+2}\Big] = \frac{2}{\lambda\gamma}\mathbb{E}\big[\Lambda_1 - \Lambda_{n+2}\big] \leq \frac{2}{\lambda\gamma}.\end{aligned}$$

The Lemma follows by passing to the limit with $n\to\infty$. ∎

7. Convergence to $W^{2,p}$ solutions: Case $f \in \mathcal{C}(\bar{\mathcal{D}})$

In this section we complete the proof of Theorem 1.1 (ii). The key point is an estimate of the remainder in the Taylor expansion (1.1), for u that is a $W^{2,p}$ solution of (RL). In the boundary layer where $\operatorname{dist}(x_0,\partial\mathcal{D}) < \epsilon$, this quantity is treated by means of Lemma 6.2, whereas in the interior of $\mathcal{D}$ we find a representation of the Newtonian potential component of u via a convolution with suitable probabilistic kernels.

Theorem 7.1. *Assume (BH) and let $u\in\mathcal{C}^1(\bar{\mathcal{D}})$ be the unique $W^{2,p}$ solution to (RL). Define the following uniformly bounded sequence of Borel "remainder" functions $\{R_\epsilon\}_{\epsilon\to 0}$:*

$$R_\epsilon(x) = u(x) - \big(1-\gamma s_\epsilon(x)\big)\fint_{B_\epsilon(x)\cap\mathcal{D}} u(y)\,\mathrm{d}y - \frac{\epsilon^2}{2(N+2)}f(x).$$

Then, for every $\epsilon\ll 1$ and $x_0\in\bar{\mathcal{D}}$ we have:

(i) $|R_\epsilon(x_0)|\leq \epsilon a_\epsilon$, *where* $\lim_{\epsilon\to 0} a_\epsilon = 0$, *uniformly in* $x_0\in\bar{\mathcal{D}}$.

(ii) $u(x_0) - \bar{u}^\epsilon(x_0) = \lim_{n\to\infty}\mathbb{E}\Big[\sum_{i=0}^{n}(R_\epsilon\circ X_i^{\epsilon,x_0})\mathbb{1}_{\{d_\epsilon(X_i^{x_0})\geq 1\}}\Lambda_i^{\epsilon,x_0}\Big]$.

(iii) *There exists a family of positive Borel functions $\{h_\epsilon : B_\epsilon(0)\to\mathbb{R}\}_{\epsilon\to 0}$ that are probability densities: $\int_{B_\epsilon(0)} h_\epsilon(y)\,\mathrm{d}y = 1$, and such that whenever dist$(x_0,\partial\mathcal{D})\geq\epsilon$, there holds:*

$$R_\epsilon(x_0) = \frac{\epsilon^2}{2(N+2)}\Big(\fint_{B_\epsilon(x_0)\cap\mathcal{D}} h_\epsilon(x_0-y)f(y)\,\mathrm{d}y - f(x_0)\Big).$$

Proof. **1.** Clearly, the Taylor expansion: $u(y) = u(x_0) + \langle\nabla u(x_0), y-x\rangle + o(\epsilon)$ holds uniformly in $y\in B_\epsilon(x_0)\cap\bar{\mathcal{D}}$ and $x_0\in\bar{\mathcal{D}}$. Arguing as in the proof of Theorem 3.1 it follows that:

$$\begin{aligned}\fint_{B_\epsilon(x_0)\cap\mathcal{D}} u(y)\,\mathrm{d}y &= u(x_0) + \Big\langle\nabla u(x_0), \fint_{B_\epsilon(x)\cap\mathcal{D}} y - x_0\,\mathrm{d}y\Big\rangle + o(\epsilon) \\ &= u(x_0) + \gamma s_\epsilon(x)u(x_0) + O(\epsilon s_\epsilon(x_0)) + o(\epsilon).\end{aligned}$$

Consequently, we obtain the bound in (i):

$$R_\epsilon(x_0) = u(x_0) - \big(1-\gamma^2 s_\epsilon(x_0)^2\big)u(x_0) + O(\epsilon s_\epsilon(x_0)) + o(\epsilon) = o(\epsilon).$$

2. To show (ii), observe that the following sequence of random variables $\{M_n\}_{n=0}^\infty$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$:

$$M_n = (u\circ X_n)\Lambda_n + \frac{\epsilon^2}{2(N+2)}\sum_{i=0}^{n-1}(f\circ X_i)\Lambda_i + \sum_{i=0}^{n-1}(R_\epsilon\circ X_i)\Lambda_i,$$

where we adopt the convention that $M_0 = 0$. Indeed:

$$\begin{aligned}\mathbb{E}\big(M_{n+1}-M_n \mid \mathcal{F}_n\big) &= \mathbb{E}\Big(u\circ X_{n+1} \mid \mathcal{F}_n\Big)\Lambda_{n+1} - (u\circ X_n)\Lambda_n + \frac{\epsilon^2}{2(N+2)}(f\circ X_n)\Lambda_n + (R_\epsilon\circ X_n)\Lambda_n \\ &= \Big(\big(1-\gamma s_\epsilon(X_n)\big)\fint_{B_\epsilon(X_n)\cap\mathcal{D}} u(y)\,\mathrm{d}y - u\circ X_n + \frac{\epsilon^2}{2(N+2)}(f\circ X_n) + R_\epsilon\Big)\Lambda_n \\ &= 0 \qquad \mathbb{P}-\text{a.s. in } \Omega.\end{aligned}$$

The above yields $u(x_0) = \mathbb{E}\big[M_0\big] = \mathbb{E}\big[M_{n+1}\big]$, resulting in:

$$u(x_0) - \frac{\epsilon^2}{2(N+2)}\mathbb{E}\Big[\sum_{i=0}^n (f\circ X_i^{\epsilon,x_0})\Lambda_i^{\epsilon,x_0}\Big] = \mathbb{E}\Big[(u\circ X_{n+1}^{\epsilon,x_0})\Lambda_{n+1}^{\epsilon,x_0}\Big] + \mathbb{E}\Big[\sum_{i=0}^n (R_\epsilon\circ X_i)\Lambda_i\Big].$$

Passing to the limit with $n\to\infty$ and using (6.3) and (6.4) implies:

$$\begin{aligned}u(x_0) - \bar{u}^\epsilon(x_0) &= \lim_{n\to 0}\mathbb{E}\Big[\sum_{i=0}^n \big(R_\epsilon\circ X_i^{\epsilon,x_0}\big)\Lambda_i\Big] \\ &= \lim_{n\to 0}\mathbb{E}\Big[\sum_{i=0}^n O\big(\epsilon a_\epsilon\big)\mathbb{1}_{\{d_\epsilon(X_i)<1\}}\Lambda_i\Big] + \lim_{n\to\infty}\mathbb{E}\Big[\sum_{i=0}^n \big(R_\epsilon\circ X_i\big)\mathbb{1}_{\{d_\epsilon(X_i)\geq 1\}}\Lambda_i\Big]\end{aligned}$$

and we conclude (ii) by Lemma 6.2.

3. Recall that the Newtonian potential $\Gamma * f\in\mathcal{C}^1(\mathbb{R}^N)$, namely $(\Gamma*f)(x) = \int_{\mathcal{D}}\Gamma(x-y)f(y)\,\mathrm{d}y$, is given in terms of the fundamental solution:

$$\Gamma(x) = \begin{cases}\frac{1}{N(N-2)|B_1^N|}|x|^{2-N} & \text{when } N>2\\ -\frac{1}{2\pi}\log|x| & \text{when } N=2,\end{cases}$$

and it satisfies: $-\Delta(\Gamma*f) = f$ in $\mathcal{D}$, in the sense of distributions [12, section 4.1]. Writing:

$$u = (\Gamma*f)_{|\bar{\mathcal{D}}} + w,$$

where $w\in\mathcal{C}^1(\bar{\mathcal{D}})$ is harmonic in $\mathcal{D}$, we derive the below formula, valid for $x_0\in\mathcal{D}$ with $\mathrm{dist}(x_0,\partial\mathcal{D})\geq\epsilon$, in virtue of the mean value property of w and by changing the integration order in convolution:

$$\begin{aligned}u(x_0) - \fint_{B_\epsilon(x_0)\cap\mathcal{D}} u(y)\,\mathrm{d}y &= (\Gamma*f)(x_0) - \fint_{B_\epsilon(x_0)}(\Gamma*f)(y)\,\mathrm{d}y \\ &= \int_{\mathcal{D}}\Big(\Gamma(x_0-y) - \fint_{B_\epsilon(x_0-y)}\Gamma(z)\,\mathrm{d}z\Big)f(y)\,\mathrm{d}y = \frac{\epsilon^2}{2(N+2)}\int_{\mathcal{D}} h_\epsilon(x_0-y)f(y)\,\mathrm{d}y.\end{aligned}$$

For the final convolution kernel, we have denoted:

$$h_\epsilon(x) = \frac{2(N+2)}{\epsilon^2}\Big(\Gamma(x) - \fint_{B_\epsilon(x)}\Gamma(y)\,\mathrm{d}y\Big).$$

The assertion (iii) follows provided we show the claimed properties of the sequence $\{h_\epsilon\}_{\epsilon\to 0}$. When $|x|\geq\epsilon$, then Γ is harmonic on $B_\epsilon(x)$, so that $h_\epsilon(x) = 0$ by the mean value property. On the other hand, for $|x|<\epsilon$, we get:

$$\begin{aligned}\int_{B_\epsilon(x)}\Gamma(y)\,\mathrm{d}y &= \int_{B_{|x|}(0)}\Gamma(x-z)\,\mathrm{d}z + \int_{|x|}^\epsilon\int_{\partial B_r(0)}\Gamma(x-z)\,\mathrm{d}\sigma(z)\,\mathrm{d}r \\ &= |B_1^N|\cdot|x|^N\Gamma(x) + \int_{|x|}^\epsilon |\partial B_1^N|\cdot r^{N-1}\fint_{\partial B_r(0)}\Gamma(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}r.\end{aligned} \tag{7.1}$$

The first term in (7.1) results by the mean value property of the harmonic function Γ on $B_{|x|}(x)$. The second term follows by noting that $g(x) = \fint_{\partial B_r(0)} \Gamma(x-y)\,\mathrm{d}\sigma(y)$ is constant on $B_r(0)$. This fact is the Newton shell theorem, that can be proved as follows. The function g is clearly harmonic on $B_r(0)$ and it is also radially symmetric: $g(x) = g(|x|)$ by the radial symmetry of Γ. Hence, the spherical mean value property yields: $g(0) = \fint_{B_s(0)} g(y)\,\mathrm{d}y = g(y_0)$ for all $|y_0| = s < r$.

In conclusion, we obtain:

$$\int_{B_\epsilon(x)} \Gamma(y)\,\mathrm{d}y = \begin{cases} \frac{1}{N(N-2)}\Big(|x|^2 + \int_{|x|}^{\epsilon} Nr\,\mathrm{d}r\Big) = \frac{\epsilon^2}{2(N-2)} - \frac{|x|^2}{2N} & \text{when } \ N>2 \\ -\frac{1}{2}\Big(|x|^2\log|x| + \int_{|x|}^{\epsilon} 2r\log|r|\,\mathrm{d}r\Big) = \frac{1}{2}\Big(\frac{\epsilon^2}{2} - \epsilon^2\log|\epsilon| - \frac{|x|^2}{2}\Big) & \text{when } \ N=2, \end{cases}$$

and further:

$$h_\epsilon(x) = \begin{cases} \frac{2(N+2)}{N(N-2)|B_1^N|}\Big(\frac{|x|^{2-N}}{\epsilon^2} - \frac{N}{2\epsilon^N} + \frac{(N-2)|x|^2}{2\epsilon^{N-2}}\Big) & \text{when } \ N>2 \\ \frac{4}{\pi}\Big(\frac{\log|\epsilon|}{\epsilon^2} + \frac{|x|^2}{2\epsilon^4} - \frac{1}{2\epsilon^2} - \frac{\log|x|}{\epsilon^2}\Big) & \text{when } \ N=2. \end{cases}$$

It follows by a straightforward calculation that $h_\epsilon(x) > 0$ for all $x \in B_\epsilon(0)$ and $\int_{B_\epsilon} h_\epsilon = 1$. ■

We are now ready to prove the remaining part of Theorem 1.1.

Corollary 7.2. *Assume (BH) and let $f \in \mathcal{C}(\bar{\mathcal{D}})$. Then $\{\bar{u}^\epsilon\}_{\epsilon\to 0}$ converge uniformly on $\bar{\mathcal{D}}$ to u that is the unique $W^{2,p}$ solution of (RL).*

Proof. By Lemma 7.1 (iii) we get, for all $x_0 \in \mathcal{D}$ with $\mathrm{dist}(x_0, \partial\mathcal{D}) \geq \epsilon$:

$$|R_\epsilon(x_0)| = o(\epsilon^2),$$

where the Landau symbol o is uniform in x_0 but may depend on f and N. Further:

$$|u(x_0) - \bar{u}_\epsilon(x_0)| \leq \lim_{n\to\infty} \mathbb{E}\Big[\sum_{i=0}^{n} |R_\epsilon \circ X_i^{\epsilon,x_0}| \mathbb{1}_{\{d_\epsilon(X_i^{x_0})\geq 1\}} \Lambda_i^{\epsilon,x_0}\Big] = o(1)$$

in view of Lemma 7.1 (ii) and (6.3). The proof is done. ■

8. The lower bound

In this section, we prove Theorem 1.2 and the optimality of the inverse quadratic estimate in Corollary 5.2, through a precise lower bound on the stopping time τ^{ϵ,x_0}. As a consequence, we obtain a uniform bound for solutions u_ϵ to $(\mathrm{RMV})_\epsilon$ with $f \geq 0$, yielding the lower bound for solutions u of the Robin problem (RL). The same optimal bound will be deduced directly for u via analytical arguments in Lemma 8.4.

Definition 8.1. *Let $\mathcal{D}$ be as in (BH) and fix a radius $\rho > 0$ that is strictly smaller than some uniform inner supporting sphere radius r of $\mathcal{D}$, given by the property:*

$$\text{for every } \ x \in \partial\mathcal{D} \ \text{ exists } \ B_r(a) \subset \mathcal{D} \quad \text{such that} \quad |x-a| = r. \tag{8.1}$$

We define a continuous function $Z^\rho : \bar{\mathcal{D}} \to \mathcal{D}$ satisfying:

$$x \in \bar{B}_\rho\big(Z^\rho(x)\big) \subset \bar{\mathcal{D}} \qquad \text{for all } \ x \in \bar{\mathcal{D}},$$

as follows. For every $x \in \bar{\mathcal{D}}$ such that $\mathrm{dist}(x, \partial\mathcal{D}) \leq \rho$, there exists exactly one $Z^\rho(x) \in \mathcal{D}$ that is the center of the inner ρ-supporting sphere at $\pi_{\partial\mathcal{D}} x$. For $x \in \mathcal{D}$ with $\mathrm{dist}(x, \partial\mathcal{D}) > \rho$, we set $Z^\rho(x) = x$. It is straightforward that for every $\epsilon \ll 1$, $x_0 \in \bar{\mathcal{D}}$ and $n \geq 0$ there holds:

$$\big|Z^\rho(X_{n+1}^{\epsilon,x_0}) - X_{n+1}^{\epsilon,x_0}\big| \leq \big|Z^\rho(X_n^{\epsilon,x_0}) - X_{n+1}^{\epsilon,x_0}\big|. \tag{8.2}$$

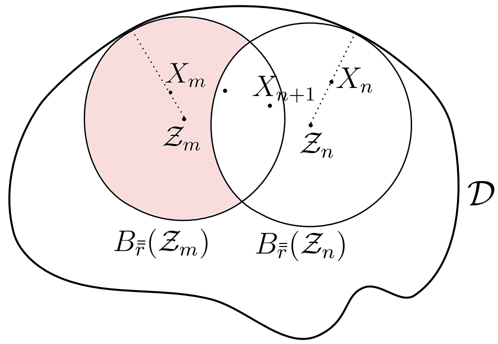

FIGURE 8.1. The auxiliary balls $B_{\bar{\bar{r}}}(\mathcal{Z}_n)$ and the process $\{X_n\}_{n=0}^\infty$ in Theorem 8.2.

Theorem 8.2. *Assume (BH) and let $\bar{r} < r$ with r as in (8.1). Then for every $\epsilon \ll 1$ there holds:*

$$\mathbb{E}\big[\tau^{\epsilon,x_0}\big] \geq \frac{2(N+2)}{\gamma N}\cdot\frac{\bar{r}}{\epsilon^2} \qquad \textit{for all } x_0 \in \bar{\mathcal{D}}. \tag{8.3}$$

Proof. **1.** Fix an intermediate radius $\bar{\bar{r}} \in (\bar{r}, r)$ and consider the auxiliary sequence of random variables $\{\mathcal{Z}_n^{\epsilon,\bar{\bar{r}},x_0} = Z^{\bar{\bar{r}}}\circ X_n^{\epsilon,x_0}\}_{n=0}^\infty$ given in Definition 8.1. As usual, we will drop the superscripts $\epsilon, \bar{\bar{r}}$ and x_0 to alleviate the notation. We now define the following sequence:

$$M_n = \big|\mathcal{Z}_n - X_n\big|^2 - n\frac{N\epsilon^2}{N+2} + \frac{2\bar{r}}{\gamma}\cdot\sum_{j=1}^n \mathbb{1}_{b_j<\gamma s_\epsilon(X_{j-1})}.$$

and aim to prove that $\{M_n\}_{n=0}^\infty$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Firstly, because of (8.2) it follows that:

$$\begin{aligned}\mathbb{E}\big(M_{n+1}-M_n \mid \mathcal{F}_n\big) &= \mathbb{E}\big(\big|\mathcal{Z}_{n+1}-X_{n+1}\big|^2 \mid \mathcal{F}_n\big) - \big|\mathcal{Z}_n - X_n\big|^2 \\ &\qquad - \frac{N\epsilon^2}{N+2} + \frac{2\bar{r}}{\gamma}\cdot\mathbb{E}\big(\mathbb{1}_{b_{n+1}<\gamma s_\epsilon(X_n)} \mid \mathcal{F}_n\big) \\ &\leq \mathbb{E}\big(\big|\mathcal{Z}_n - X_{n+1}\big|^2 \mid \mathcal{F}_n\big) - \big|\mathcal{Z}_n - X_n\big|^2 - \frac{N\epsilon^2}{N+2} + 2\bar{r}\cdot s_\epsilon(X_n).\end{aligned} \tag{8.4}$$

Then, with the help of (2.3), (2.5) and (5.1) we get:

$$\begin{aligned}&\mathbb{E}\big(\big|\mathcal{Z}_n - X_{n+1}\big|^2 \mid \mathcal{F}_n\big) - \big|\mathcal{Z}_n - X_n\big|^2 \\ &\qquad = \fint_{B_\epsilon(0)} \Big|\mathcal{Z}_n - \Big(X_n + \epsilon w_n^{X_{n-1}}\Big)\Big|^2 \,\mathrm{d}w_{n+1} - \big|\mathcal{Z}_n - X_n^{x_0}\big|^2 \\ &\qquad = \fint_{B_\epsilon(X_n^{x_0})\cap\mathcal{D}} |y-\mathcal{Z}_n|^2\,\mathrm{d}y - \big|\mathcal{Z}_n - X_n\big|^2 \\ &\qquad = \fint_{B_\epsilon(X_n)\cap\mathcal{D}} |y-X_n|^2\,\mathrm{d}y + 2\Big\langle \fint_{B_\epsilon(X_n)\cap\mathcal{D}} y - X_n\,\mathrm{d}y, X_n - \mathcal{Z}_n\Big\rangle \\ &\qquad = \frac{N\epsilon^2}{N+2} - 2s_\epsilon(X_n)\big\langle\vec{n}(\pi_{\partial\mathcal{D}}X_n), X_n - \mathcal{Z}_n\big\rangle + \mathcal{O}(\epsilon s_\epsilon(X_n)).\end{aligned}$$

Since $X_n - \mathcal{Z}_n = \big|X_n - \mathcal{Z}_n\big| \cdot \vec{n}(\pi_{\partial\mathcal{D}} X_n)$, and since $|X_n - \mathcal{Z}_n| \geq \bar{\bar{r}} - \epsilon$ when $s_\epsilon(X_n) \neq 0$, it follows that for all $\epsilon \ll \bar{\bar{r}} - \bar{r}$ we get:

$$\mathbb{E}\big(\big|\mathcal{Z}_n - X_{n+1}\big|^2 \mid \mathcal{F}_n\big) - \big|\mathcal{Z}_n - X_n\big|^2 = \frac{N\epsilon^2}{N+2} - 2s_\epsilon(X_n) \cdot \big|X_n - \mathcal{Z}_n\big| + \mathcal{O}(\epsilon s_\epsilon(X_n))$$
$$\leq \frac{N\epsilon^2}{N+2} - 2\bar{\bar{r}} s_\epsilon(X_n) + \mathcal{O}(\epsilon s_\epsilon(X_n)) \leq \frac{N\epsilon^2}{N+2} - 2\bar{r} s_\epsilon(X_n).$$

Recalling (8.4), we conclude the desired supermartingale property: $\mathbb{E}\big(M_{n+1} - M_n \mid \mathcal{F}_n\big) \leq 0$.

2. For each fixed $k \geq 1$, apply now Doob's Optional Stopping to $\{\mathcal{Z}_n\}_{n=0}^\infty$ and the finite stopping time $\tau^{\epsilon,x_0} \wedge k$, to the effect that:

$$\bar{\bar{r}}^2 \geq |\mathcal{Z}_0 - x_0|^2 = \mathbb{E}\big[M_0\big] \geq \mathbb{E}\big[M_{\tau^{\epsilon,x_0} \wedge k}\big]$$
$$= \mathbb{E}\big[\big|\mathcal{Z}_{\tau^{\epsilon,x_0}\wedge k} - X_{\tau^{\epsilon,x_0}\wedge k}\big|^2\big] - \mathbb{E}\big[\tau^{\epsilon,x_0} \wedge k\big]\frac{N\epsilon^2}{N+2} + \frac{2\bar{r}}{\gamma} \cdot \mathbb{P}\big(\tau^{\epsilon,x_0} \leq k\big).$$

Passing to the limit with $k \to \infty$, we obtain:

$$\bar{\bar{r}}^2 \geq \mathbb{E}\big[\big|\mathcal{Z}_{\tau^{\epsilon,x_0}} - X_{\tau^{\epsilon,x_0}}\big|^2\big] - \mathbb{E}\big[\tau^{\epsilon,x_0}\big]\frac{N\epsilon^2}{N+2} + \frac{2\bar{r}}{\gamma} \geq |\bar{\bar{r}} - \epsilon|^2 - \mathbb{E}\big[\tau^{\epsilon,x_0}\big]\frac{N\epsilon^2}{N+2} + \frac{2\bar{r}}{\gamma}.$$

Writing $|\bar{\bar{r}} - \epsilon|^2 \geq \bar{\bar{r}}^2 - 2\bar{\bar{r}}\epsilon$, it follows that:

$$\mathbb{E}\big[\tau^{\epsilon,x_0}\big] \cdot \frac{N\epsilon^2}{N+2} \geq \frac{2\bar{r}}{\gamma} - 2\bar{\bar{r}}\epsilon,$$

which implies (8.3) for $\bar{r}$ replaced by any smaller radius, provided that $\epsilon \ll 1$. This completes the proof of the claim, for any $\bar{r} < r$. ■

Corollary 8.3. *Assume (BH), let $f \geq 0$ and let $r > 0$ satisfy (8.1). Then:*

(i) *The solution to $(\mathrm{RMV})_\epsilon$ satisfies:*

$$u_\epsilon(x_0) \geq \frac{\bar{r}}{\gamma N} \cdot \inf_{\bar{\mathcal{D}}} f \qquad \textit{for all} \;\; x_0 \in \bar{\mathcal{D}},$$

for any radius $\bar{r} < r$, provided that $\epsilon \ll 1$.

(ii) *If some sequence of solutions $\{u_\epsilon\}_{\epsilon\to 0}$ to $(\mathrm{RMV})_\epsilon$ converges pointwise on $\bar{\mathcal{D}}$, then the limit $u : \bar{\mathcal{D}} \to \mathbb{R}$ satisfies:*

$$u(x_0) \geq \frac{r}{\gamma N} \cdot \inf_{\bar{\mathcal{D}}} f \qquad \textit{for all} \;\; x_0 \in \bar{\mathcal{D}}. \tag{8.5}$$

Proof. From (8.3) and the definition (5.4) it directly follows that:

$$u^\epsilon(x_0) \geq \frac{\epsilon^2}{2(N+2)}\mathbb{E}\big[\tau^{\epsilon,x_0} - 1\big] \cdot \inf_{\bar{\mathcal{D}}} f \geq \Big(\frac{\bar{r}}{\gamma N} - \frac{\epsilon^2}{2(N+2)}\Big) \cdot \inf_{\bar{\mathcal{D}}} f,$$

resulting in (i) for $\bar{r}$ replaced by any smaller radius, for $\epsilon \ll 1$. Recalling Lemma 5.1, this completes the proof of the claim for any $\bar{r} < r$. The limiting statement (ii) is self-evident from (i). ■

Automatically, the limiting solution u of $\{u^\epsilon\}_{\epsilon\to 0}$ obeys the bound in (8.5). We now present an analytical proof of the same result, under stronger regularity assumptions:

Lemma 8.4. *Let $\mathcal{D} \subset \mathbb{R}^N$ be an open, bounded and connected set, satisfying the uniform inner supporting sphere condition with radius $r > 0$ as in (8.1). Given two constants: $\gamma > 0$ and $m \geq 0$, assume that $u \in \mathcal{C}^1(\bar{\mathcal{D}})$ satisfies (in the sense of distributions in $\mathcal{D}$):*

$$-\Delta u \geq m \quad \text{in } \mathcal{D}, \qquad \frac{\partial u}{\partial \vec{n}} + \gamma u = 0 \quad \text{on } \partial\mathcal{D}.$$

Then: $\min_{\bar{\mathcal{D}}} u \geq \frac{rm}{\gamma N}$.

Proof. By the maximum principle, there holds: $\min_{\bar{\mathcal{D}}} u = \min_{\partial\mathcal{D}} u = u(x_0)$ for some $x_0 \in \partial\mathcal{D}$. Consider the inner supporting ball $B = B(x_0 - r\vec{n}(x_0))$ and the function $v_r \in \mathcal{C}^2(\bar{B})$ given by:

$$v_r\big(x_0 - r\vec{n}(x_0) + y\big) = v(|y|) \quad \text{where: } v(s) = \frac{m}{2N}(r^2 - s^2) + \frac{rm}{\gamma N}.$$

Since:

$$-\Delta v_r = m \quad \text{in } B, \qquad \frac{\partial v_r}{\partial \vec{n}} + \gamma v_r = 0 \quad \text{on } \partial B,$$

it follows that: $-\Delta(u - v_r) \geq 0$. Applying again the maximum principle, we get:

$$\min_{\bar{B}}(u - v_r) = \min_{\partial B}(u - v_r) = \min_{\partial B} u - \frac{rm}{\gamma N} = u(x_0) - \frac{rm}{\gamma N}.$$

In particular, the difference $u - v_r$ is minimized at x_0, so: $\frac{\partial(u-v_r)}{\partial \vec{n}}(x_0) \leq 0$, and consequently:

$$(u - v_r)(x_0) = -\frac{1}{\gamma} \cdot \frac{\partial(u - v_r)}{\partial \vec{n}}(x_0) \geq 0.$$

In conclusion: $u(x_0) \geq v(x_0) = \frac{rm}{\gamma N}$, proving the claim. ■

Remark 8.5. The bound in Lemma 8.4 is optimal. Take $\mathcal{D} = B_r(0)$ and $u(x) = a - |x|^2$. Then:

$$-\Delta u = 2N \quad \text{in } \mathcal{D}, \qquad \frac{\partial u}{\partial \vec{n}} + \gamma u = -2r + \gamma(a - r^2) \quad \text{on } \partial\mathcal{D},$$

so (RL) holds with $f = 2N$ and $\gamma = \frac{2r}{a-r^2} > 0$ by taking $a > r^2$. Then, clearly:

$$\min_{\bar{\mathcal{D}}} u = a - r^2 = \frac{r}{2rN}(a - r^2)2N = \frac{r}{\gamma N}\min_{\bar{\mathcal{D}}} f.$$

We also remark that using the arguments in [5, Lemma 3.2], [22], one can prove the lower bound on u involving the integral $\int_{\mathcal{D}} f(y)\,\mathrm{d}y$, rather than the pointwise quantity $\min_{\bar{\mathcal{D}}} f$.

9. Appendix: A proof of uniqueness of viscosity solutions to (RL)

We first make an observation that relies on the assumed regularity of $\partial\mathcal{D}$.

Lemma 9.1. *When $\mathcal{D} \subset \mathbb{R}^N$ satisfies the uniform outer supporting sphere condition:*

$$\text{for every } x \in \partial\mathcal{D} \text{ exists } B_r(b) \subset \mathbb{R}^n \setminus \bar{\mathcal{D}} \quad \text{such that} \quad |x - b| = r, \tag{9.1}$$

with some radius $r > 0$, then the boundary requirements in Definition 4.1 (i) and (ii) can be reduced to: $\langle p, \vec{n}(x)\rangle + \gamma u(x) \leq 0$ and $\langle p, \vec{n}(x)\rangle + \gamma u(x) \geq 0$, respectively.

Proof. Let $x \in \partial\mathcal{D}$ and $(p, X) \in J^{2,+}_{\bar{\mathcal{D}}} u(x)$. For each large j, consider the jet:

$$(p_j, X_j) = \Big(p - \frac{1}{j}\vec{n}(x),\ X + rId_N - (r+j)\vec{n}(x)^{\otimes 2}\Big).$$

We claim that $(p_j, X_j) \in J^{2,+}_{\bar{\mathcal{D}}} u(x)$. In this case, we have:

$$-\mathrm{trace} X_j = -\big(\mathrm{trace} X + (N-1)r - j\big) > f(x),$$

so by (i) there must be: $0 \geq \langle p_j, \vec{n}(x)\rangle + \gamma u(x) = \langle p, \vec{n}(x)\rangle + \gamma u(x) - \frac{1}{j}$. Passing to the limit with $j \to \infty$ we get the claimed boundary condition. To show that (p_j, X_j) is indeed a super-jet, let:

$$\begin{aligned} q_j(y-x) &= \Big\langle -\frac{1}{j}\vec{n}(x), y-x\Big\rangle + \frac{1}{2}\Big\langle rId_N - (r+j)\vec{n}(x)^{\otimes 2} : (y-x)^{\otimes 2}\Big\rangle \\ &= -\frac{1}{j}\langle \vec{n}(x), y-x\rangle + \frac{r}{2}|(y-x)_{tan}|^2 - \frac{j}{2}\langle \vec{n}(x), y-x\rangle^2. \end{aligned}$$

For $y \in \bar{\mathcal{D}}$ satisfying $\langle \vec{n}(x), y-x\rangle \leq 0$, we get: $q_j(y-x) \geq |\langle \vec{n}(x), y-x\rangle|\Big(\frac{1}{j} - \frac{j}{2}|\langle \vec{n}(x), y-x\rangle|\Big) \geq 0$, if $|y-x|$ is small enough. On the other other hand, for $y \in \bar{\mathcal{D}}$ such that $\langle \vec{n}(x), y-x\rangle \geq 0$, we get:

$$\begin{aligned} q_j(y-x) &\geq -\frac{2}{j}\langle \vec{n}(x), y-x\rangle + \frac{r}{2}|(y-x)_{tan}|^2 \\ &\geq -\frac{2}{j}\Big(r - \sqrt{r^2 - |(y-x)_{tan}|^2}\Big) + \frac{r}{2}|(y-x)_{tan}|^2 \geq 0, \end{aligned}$$

as for small $|y-x|$ and $j \gg 1$ there holds: $r - \sqrt{r^2 - |(y-x)_{tan}|^2} \leq \frac{1}{r}|(y-x)_{tan}|^2 \leq \frac{jr}{4}|(y-x)_{tan}|^2$. Thus, in both cases, the validity of (4.1) for (p, X), implies the same asymptotic bound for each (p_j, X_j) with sufficiently large j. ∎

Lemma 9.2. *Assume the uniform outer supporting sphere condition (9.1) with radius $r > 0$. Then the Robin problem (RL) with $f \in \mathcal{C}(\bar{\mathcal{D}})$ has at most one viscosity solution.*

Proof. **1.** Let u, v be two viscosity solutions to (RL). We will prove that $u \leq v$. In fact, the same analysis works when u is a viscosity sub-solution and v is a viscosity super-solution, in the sense of Definition 4.1 (i) and (ii), where u is assumed only to be upper-semicontinuous and v lower-semicontinuous, and where the jets sets $J^{2,+}_{\bar{\mathcal{D}}}$ and $J^{2,-}_{\bar{\mathcal{D}}}$ are replaced by their closures $\bar{J}^{2,+}_{\bar{\mathcal{D}}}$ and $\bar{J}^{2,-}_{\bar{\mathcal{D}}}$, respectively (see [9] for the details). Also, we recall that the requirements at boundary points in Definition 4.1 can be reduced as in Lemma 9.1, because of (9.1). The stated comparison principle is proved in three steps: by replacing u and v with strict sub- and super-solutions u_δ and v_δ, and by doubling the variables technique with an appropriate nonlinear corrector, separately in the cases when u_δ and v_δ achieves its maximum in $\mathcal{D}$ or on $\partial\mathcal{D}$. We now sketch these arguments.

For a sufficiently large $C \gg 1$ and each $\delta \ll 1$, define:

$$u_\delta(x) = u(x) - \frac{\delta}{2}\big(|x|^2 - C\big), \qquad v_\delta(x) = v(x) + \frac{\delta}{2}\big(|x|^2 - C\big).$$

Assume that $(p, X) \in J^{2,+}_{\bar{\mathcal{D}}} u_\delta(x)$, which is equivalent to: $(p - \delta x, X - \delta Id_N) \in J^{2,+}_{\bar{\mathcal{D}}} u(x)$. Then:

$$\begin{aligned} &- \mathrm{trace}\, X \leq f(x) - N\delta && \text{when } x \in \mathcal{D} \\ &\langle p, \vec{n}(x)\rangle + \gamma u_\delta(x) \leq -\delta && \text{when } x \in \partial\mathcal{D}. \end{aligned} \tag{9.2}$$

Thus, each u_δ is a strict sub-solution of (RL). Similarly, each v_δ is a strict super-solution, namely $(p,X)\in J^{2,-}_{\bar{\mathcal{D}}}v_\delta(x)$ implies:

$$\begin{aligned} -\operatorname{trace} X &\geq f(x)+N\delta \qquad &&\text{when } x\in\mathcal{D}\\ \langle p,\vec{n}(x)\rangle+\gamma v_\delta(x)&\geq\delta &&\text{when } x\in\partial\mathcal{D}. \end{aligned}\tag{9.3}$$

2. We will show that $u_\delta\leq v_\delta$ in $\bar{\mathcal{D}}$, for all $\delta\ll 1$. By contradiction, fix $\delta>0$ and assume that:

$$\max_{\bar{\mathcal{D}}}\big(u_\delta-v_\delta\big)>0.$$

We first treat the case of $\max_{\bar{\mathcal{D}}}\big(u_\delta-v_\delta\big)>\max_{\partial\bar{\mathcal{D}}}\big(u_\delta-v_\delta\big)$. We apply [9, Proposition 3.7] to: $\Phi(x,y)=u_\delta(x)-v_\delta(y)$ and $\Psi(x,y)=\frac{1}{2}|x-y|^2$ and obtain a sequence $\{(x_\alpha,y_\alpha)\}_{\alpha\to\infty}$ of maximizers to $\Phi-\alpha\Psi$ on $\bar{\mathcal{D}}\times\bar{\mathcal{D}}$ that converges to some diagonal element (z_0,z_0) such that z_0 is a maximizer of $u_\delta-v_\delta$ and thus $z_0\in\mathcal{D}$. Applying [9, Theorem 3.2] to: $w(x,y)=u_\delta(x)-v_\delta(y)$ and $\phi(x,y)=\frac{\alpha}{2}|x-y|^2$, we further obtain sequences of matrices $X_\alpha,Y_\alpha\in\mathbb{R}^{N\times N}_{\text{sym}}$ satisfying:

$$\begin{aligned} &\big(\alpha(x_\alpha-y_\alpha),X_\alpha\big)\in J^{2,+}_{\bar{\mathcal{D}}}u_\delta(x_\alpha),\qquad \big(\alpha(x_\alpha-y_\alpha),Y_\alpha\big)\in J^{2,-}_{\bar{\mathcal{D}}}v_\delta(y_\alpha)\\ &\begin{bmatrix} X_\alpha & 0\\ 0 & -Y_\alpha\end{bmatrix}\leq 3\alpha\begin{bmatrix} Id_N & -Id_N\\ -Id_N & Id_N\end{bmatrix}. \end{aligned}\tag{9.4}$$

The first two assertions above, together with (9.2), (9.3) yield:

$$-\text{trace}\,(X_\alpha-Y_\alpha)\leq f(x_\alpha)-f(y_\alpha)-2N\delta\to-2\delta\quad\text{ as }\ \alpha\to\infty,$$

which contradicts the last assertion in (9.4) that implies: trace $(X_\alpha-Y_\alpha)\leq 0$.

3. We now treat the remaining case, namely that of:

$$\max_{\bar{\mathcal{D}}}\big(u_\delta-v_\delta\big)=\big(u_\delta-v_\delta\big)(z_0)>0\quad\text{ for some }\ z_0\in\partial\mathcal{D}.\tag{9.5}$$

Applying [9, Proposition 3.7] to:

$$\Phi(x,y)=u_\delta(x)-v_\delta(y)-\gamma u_\delta(z_0)\langle y-x,\vec{n}(z_0)\rangle-|x-z_0|^2,\qquad \Psi(x,y)=\frac{1}{2}|x-y|^2,$$

we obtain a sequence $\{(x_\alpha,y_\alpha)\}_{\alpha\to\infty}$ of maximizers to $\Phi-\alpha\Psi$ that converges to some (z,z), where z is a maximizer of $\Phi(z,z)=u_\delta(z)-v_\delta(z)-|z-z_0|^4$. Hence there must be $z=z_0$. Also:

$$\alpha|x_\alpha-y_\alpha|^2\to 0\quad\text{ as }\alpha\to\infty.\tag{9.6}$$

We now apply [9, Theorem 3.2] to:

$$w(x,y)=u_\delta(x)-v_\delta(y),\qquad \phi(x,y)=\frac{\alpha}{2}|x-y|^2+\gamma u_\delta(z_0)\langle y-x,\vec{n}(z_0)\rangle,$$

which yields existence of sequences of matrices $X_\alpha,Y_\alpha\in\mathbb{R}^{N\times N}_{\text{sym}}$ satisfying:

$$\begin{aligned} &\big(\alpha(x_\alpha-y_\alpha)-\gamma u_\delta(z_0)\vec{n}(z_0)+4|x_\alpha-z_0|^2(x_\alpha-z_0),X_\alpha\big)\in J^{2,+}_{\bar{\mathcal{D}}}u_\delta(x_\alpha),\\ &\big(\alpha(x_\alpha-y_\alpha)-\gamma u_\delta(z_0)\vec{n}(z_0),Y_\alpha\big)\in J^{2,-}_{\bar{\mathcal{D}}}v_\delta(y_\alpha),\\ &\begin{bmatrix} X_\alpha & 0\\ 0 & -Y_\alpha\end{bmatrix}\leq\nabla^2\phi(x_\alpha,y_\alpha)+\frac{1}{\alpha}\nabla^2\phi(x_\alpha,y_\alpha)^2. \end{aligned}\tag{9.7}$$

Since $\nabla^2\phi(x_\alpha,y_\alpha)=\alpha\begin{bmatrix} Id_N & -Id_N\\ -Id_N & Id_N\end{bmatrix}+\mathcal{O}\big(|x_\alpha-z_0|^2\big)$, the last assertion above implies:

$$\text{trace}\,(X_\alpha-Y_\alpha)\leq\mathcal{O}\big(|x_\alpha-z_0|^2+\frac{1}{\alpha}|x_\alpha-z_0|^4\big)\quad\to 0\quad\text{ as }\alpha\to\infty.\tag{9.8}$$

Note that for large $\alpha \gg 1$ there must be $x_\alpha, y_\alpha \in \mathcal{D}$. Indeed, if $x_\alpha \in \partial\mathcal{D}$ then (9.2) and the first assertion in (9.7) yield the contradiction in:

$$\begin{aligned}-\delta &\geq \alpha\langle x_\alpha - y_\alpha, \vec{n}(x_\alpha)\rangle - \gamma u_\delta(z_0)\langle \vec{n}(z_0), \vec{n}(x_\alpha)\rangle + 4|x_\alpha - z_0|^2\langle x_\alpha - z_0, \vec{n}(x_\alpha)\rangle + \gamma u_\delta(x_\alpha)\\ &\geq -\frac{\alpha}{2r}|x_\alpha - y_\alpha|^2 + \gamma u_\delta(z_0)\langle \vec{n}(z_0), \vec{n}(x_\alpha)\rangle + \mathcal{O}\big(|x_\alpha - z_0|^3\big) + \gamma u_\delta(x_\alpha)\\ &\to 0 \quad \text{ as } \alpha \to \infty,\end{aligned}$$

where we used (9.1) for the bound $\langle y_\alpha - x_\alpha, \vec{n}(x_\alpha)\rangle \leq \frac{1}{2r}|x_\alpha - y_\alpha|^2$, followed by (9.8). Similarly, if $y_\alpha \in \partial\mathcal{D}$ then (9.3) and the second assertion in (9.7) brings the contradiction with (9.5), as:

$$\begin{aligned}\delta &\leq \alpha\langle x_\alpha - y_\alpha, \vec{n}(y_\alpha)\rangle - \gamma u_\delta(z_0)\langle \vec{n}(z_0), \vec{n}(y_\alpha)\rangle + \gamma v_\delta(y_\alpha)\\ &\leq \frac{\alpha}{2r}|x_\alpha - y_\alpha|^2 - \gamma u_\delta(z_0)\langle \vec{n}(z_0), \vec{n}(y_\alpha)\rangle + \gamma v_\delta(y_\alpha) \to \gamma\big(v_\delta(z_0) - u_\delta(z_0)\big) \quad \text{ as } \alpha \to \infty.\end{aligned}$$

The fact of $x_\alpha, y_\alpha \in \mathcal{D}$ established, we use (9.7) together with (9.2), (9.3), to obtain:

$$-\text{trace}\,\big(X_\alpha - Y_\alpha\big) \leq f(x_\alpha) - f(y_\alpha) - 2N\delta \to -2N\delta \quad \text{ as } \alpha \to \infty,$$

contradicting (9.8). This ends the proof of the Lemma. ■

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