HARMONIC ANALYSIS

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1. Test functions and distributions

Let $\Omega$ be a nonempty open set in $\mathbb{R}^n$. The test function space $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of all infinitely differentiable functions with compact support.

For every compact $K \subset \Omega$, we use $\mathcal{D}_K(\Omega) \subset \mathcal{D}(\Omega)$ to define the collection of all infinitely differentiable functions with compact support in $K$. We can introduce a metric in $\mathcal{D}_K(\Omega)$ such that for any $f, g \in \mathcal{D}_K(\Omega)$,

$$d_K(f, g) = \max_{0 \leq N < \infty} \frac{1}{2^N} \frac{\|f - g\|_N}{1 + \|f - g\|_N}$$

where

$$\|f\|_N = \max_{|\alpha| \leq N} \|D^\alpha f\|_{L^\infty}, \ N \geq 0.$$ 

We use $\tau_K$ to denote the topology of $\mathcal{D}_K(\Omega)$ equipped with such metric.

The topology of $\mathcal{D}(\Omega)$ can be defined precisely. Let $\beta$ be the collection of all convex balanced sets $W \subset \mathcal{D}(\Omega)$ such that $\mathcal{D}_K(\Omega) \cap W \in \tau_K$ for every compact $K \subset \Omega$. Let $\tau$ be the collection of all unions of sets of the form $\phi + W$ with $\phi \in \mathcal{D}(\Omega)$ and $W \in \beta$.

**Theorem 1.** $\tau$ is a topology in $\mathcal{D}(\Omega)$ and $\beta$ is a local base for $\tau$. The topology $\tau$ makes $\mathcal{D}(\Omega)$ into a locally convex topological vector space.

Many important properties of $\mathcal{D}(\Omega)$ are included in the following theorem.

**Theorem 2.** (a) A convex balanced subset $V$ of $\mathcal{D}(\Omega)$ is open iff $V \in \beta$.
(b) The topology $\tau_K$ of any $\mathcal{D}_K(\Omega)$ coincides with the subspace topology that $\mathcal{D}_K$ inherits from $\mathcal{D}(\Omega)$.
(c) If $E$ is a bounded subset of $\mathcal{D}(\Omega)$, then $E \subset \mathcal{D}_K(\Omega)$ for some compact $K \subset \Omega$ and for each $N \geq 0$, there exists $M_N < \infty$ s.t.,

$$\|\phi\|_N \leq M_N \text{ for any } \phi \in E.$$

(d) $\mathcal{D}(\Omega)$ has the Heine-Borel property.
(e) If $\{\phi_i\}$ is a Cauchy sequence in $\mathcal{D}(\Omega)$, then $\{\phi_i\} \subset \mathcal{D}_K(\Omega)$ for some compact $K \subset \Omega$ and

$$\lim_{i,j \to \infty} \|\phi_i - \phi_j\|_N = 0 \text{ for any } N \geq 0.$$

(f) If $\phi_i \to 0$, then $\{\phi_i\} \subset \mathcal{D}_K(\Omega)$ for some compact $K \subset \Omega$ and $D^\alpha \phi_i \to 0$ uniformly for every multi-index $\alpha$.
(g) In $\mathcal{D}(\Omega)$, every Cauchy sequence converges.

**Theorem 3.** Every differential operator $D^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is continuous.

**Definition 1.** A continuous linear functional on $\mathcal{D}(\Omega)$ is called a distribution. The space of all distributions in $\Omega$ is denoted by $\mathcal{D}'(\Omega)$. 

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Theorem 4. If $\Lambda$ is a linear functional on $\mathcal{D}(\Omega)$, the following two conditions are equivalent:

(a) $\Lambda \in \mathcal{D}'(\Omega)$.

(b) For every compact $K \subset \Omega$, there exists $N \geq 0$ and $C > 0$ s.t.

$$|\Lambda \phi| \leq C \|\phi\|_N$$
holds for every $\phi \in \mathcal{D}_K(\Omega)$.

Remark 1. If $\Lambda$ is such that one $N$ will do for all $K$, then the smallest such $N$ is called the order of $\Lambda$.

Example 1. $\delta_x$ is a distribution of order 0.

Example 2. Every locally integrable complex function is a distribution of order 0.

Example 3. Any locally finite Borel measure is a distribution of order 0.

The space of distribution $\mathcal{D}(\Omega)$ is a topological vector space with weak-* topology, and it has many other nice structures.

Definition 2. If $\alpha$ is a multi-index and $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda \in \mathcal{D}'(\Omega)$ is a linear functional defined by the formula

$$(D^\alpha \Lambda)(\phi) = (-1)^{|\alpha|} \Lambda (D^\alpha \phi), \phi \in \mathcal{D}(\Omega).$$

Proposition 1. For any multi-indices $\alpha, \beta$ and for any $\Lambda \in \mathcal{D}'(\Omega)$,

$$D^\beta D^\alpha \Lambda = D^\alpha D^\beta \Lambda = D^{\alpha + \beta} \Lambda.$$

Remark 2. If $f$ has continuous partial derivatives of all orders up to $N$, then

$$D^\alpha \Lambda f = \Lambda D^\alpha f$$
for any $|\alpha| \leq N$.

Definition 3. Suppose $\Lambda \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. We can define a distribution $f \Lambda$ such that

$$(f \Lambda)(\phi) = \Lambda (f \phi), \phi \in \mathcal{D}(\Omega).$$

Locally, distributions are derivatives of distributions induced by continuous functions:

Theorem 5. Suppose $\Lambda \in \mathcal{D}'(\Omega)$ and $K$ is a compact subset of $\Omega$. Then there is a continuous function $f$ in $\Omega$ and there is a multi-index $\alpha$ such that for every $\phi \in \mathcal{D}_K(\Omega)$,

$$\Lambda \phi = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x) \, dx.$$
i.e.,

$$\Lambda = D^\alpha f$$ on $K$.

These results on distributions are proved in Analysis 3, please check Chapter 6 of Rudin’s Functional Analysis for details.
2. Fourier Analysis

We consider complex valued periodic functions on $\mathbb{R}$ with period $2\pi$ which can be viewed as functions defined on the unit circle $S^1$.

The Fourier Coefficients of the function $f \in L^1(S^1) \equiv L^1[-\pi, \pi]$ are defined by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.$$ 

Formally, the Fourier Series of $f$ is

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$ 

Since $f \in L^1[-\pi, \pi]$, we have

$$|a_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, dx = \frac{1}{2\pi} \|f\|_{L^1}.$$ 

Moreover, we have the Riemann-Lebesgue lemma:

**Theorem 6.** For every $f \in L^1[-\pi, \pi]$,

$$\lim_{n \to \pm \infty} a_n = 0.$$ 

If $f \in C^k(S^1)$, i.e., $f \in C^k(\mathbb{R})$ is periodic with period $2\pi$, then integration by parts yields for any $n \neq 0$,

$$|a_n| \leq \frac{1}{2\pi n^k} \left\| \frac{d^k f}{dx^k} \right\|_{L^1} \leq \frac{1}{n^k} \left\| \frac{d^k f}{dx^k} \right\|_{L^\infty}.$$ 

In particular, if $f \in C^2(S^1)$, Weierstrass’s M-test implies that the Fourier Series of $f$

$$\sum_{n=-\infty}^{\infty} a_n e^{inx}$$

converges uniformly to a continuous function $g \in C(S^1)$. We will show that $g = f$ later.

To study the convergence of Fourier series, we define the partial sums

$$s_N(f,x) = \sum_{n=-N}^{N} a_n e^{inx},$$

and the Fejer sums

$$S_N(f,x) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(f,x).$$

Direct calculations yield

$$s_N(f,x) = \int_{-\pi}^{\pi} f(y) k_N(x-y) \, dy$$

where

$$k_N(z) = \frac{1}{2\pi} \frac{\sin \left( N + \frac{1}{2} \right) z}{\sin \frac{z}{2}}$$

and

$$S_N(f,x) = \int_{-\pi}^{\pi} f(y) K_N(x-y) \, dy$$
where
\[ K_N(z) = \frac{1}{\pi(N+1)} \left[ \frac{\sin(N + \frac{1}{2})z}{\sin \frac{z}{2}} \right]^2. \]

If we take \( f \equiv 1 \), then for each \( N \geq 0, s_N(f, x) \equiv S_N(f, x) \equiv 1 \) and hence
\[ \int_{-\pi}^{\pi} k_N(x) \, dx = \int_{-\pi}^{\pi} K_N(x) \, dx = 1. \]

We observe also that \( K_N(x) \geq 0 \) and for each \( 0 < \delta < \pi \), \( K_N \) converges to \( 0 \) uniformly on \( \delta \leq |x| \leq \pi \) as \( N \to \infty \) which implies
\[ \lim_{N \to \infty} \int_{\delta \leq |x| \leq \pi} K_N(x) \, dx = 0. \]

**Theorem 7.** For any \( f \in C(S^1) \), the Fejer sum \( S_N(f, x) \) converge to \( f \) uniformly on \( S^1 \).

We leave the proof as an exercise.

**Theorem 8.** For any \( f \in L^p[-\pi, \pi], p \geq 1 \), the Fejer sum \( S_N(f, x) \) converges to \( f \) in \( L^p[-\pi, \pi] \).

**Proof.** Young’s inequality implies
\[ ||S_N(f, x)||_{L^p} \leq 2^{\frac{4}{p}} ||f||_{L^p}. \]

Since \( C(S^1) \) is dense in \( L^p[-\pi, \pi] \), the result follows from 7. \( \square \)

**Corollary 1.** For any \( f \in C^2(S^1) \), the Fourier series of \( f \) converges to \( f \) uniformly.

**Proof.** Since
\[ |a_n| \leq \frac{1}{n^2} ||f''||_{L^\infty}, \]
\( s_N(f, x) \) converges uniformly, since \( S_N(f, x) \) converges uniformly to \( f \), we conclude \( s_N(f, x) \) converges uniformly to \( f \). \( \square \)

**Theorem 9.** If \( f \in C^\alpha(S^1) \) for some \( \alpha > 0 \), then for any \( x \),
\[ \lim_{N \to \infty} s_N(f, x) = f(x). \]

**Proof.**
\[ s_N(f, x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x-y) - f(x)}{\sin \frac{y}{2}} \sin \left( N + \frac{1}{2} \right) y \, dy. \]
Since
\[ \frac{f(x-y) - f(x)}{\sin \frac{y}{2}} \]
is integrable, the result follows from Riemann-Lebesgue lemma. \( \square \)

**Remark 3.** Actually, the result can be strengthened to: If \( f \in C^\alpha(S^1) \) for some \( \alpha > 0 \), then \( s_N(f, x) \) converges to \( f \) uniformly.

**Remark 4.** There exists \( f \in C(S^1) \) such that \( s_N(f, x) \) doesn’t converge to \( f \) pointwisely.
Theorem 10. If $f \in BV(S^1)$, then for any $x$,
\[
\lim_{N \to \infty} s_N(f, x) = \frac{f(x^+) + f(x^-)}{2}.
\]

Proof. We first consider
\[
\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(-y) \frac{\sin \lambda y}{y} dy.
\]
Let
\[G(y) = \int_{y}^{\infty} \frac{\sin y}{y} dy.
\]
Then we have
\[
\frac{1}{\pi} \int_{0}^{\pi} f(-y) \frac{\sin \lambda y}{y} dy
= \frac{1}{\pi} \int_{0}^{\frac{\lambda \pi}{\lambda}} f\left(-\frac{y}{\lambda}\right) \frac{\sin y}{y} dy
= -\frac{1}{\pi} \int_{0}^{\frac{\lambda \pi}{\lambda}} f\left(-\frac{y}{\lambda}\right) dG(y)
= -G\left(y\right) f\left(-\frac{y}{\lambda}\right)\big|_{0}^{\frac{\lambda \pi}{\lambda}} + \frac{1}{\pi} \int_{0}^{\frac{\lambda \pi}{\lambda}} G(y) df\left(-\frac{y}{\lambda}\right)
= -G(\lambda \pi) f(-\pi^+) + G(0) f(0^-) + \frac{1}{\pi} \int_{0}^{\pi} G(\lambda y) df(-y),
\]

hence
\[
\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{0}^{\pi} f(-y) \frac{\sin \lambda y}{y} dy = \frac{1}{2} f(0^-).
\]

Similarly
\[
\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\pi}^{0} f(-y) \frac{\sin \lambda y}{y} dy = \frac{1}{2} f(0^+).
\]

So we have
\[
\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{0}^{\pi} f(-y) \frac{\sin \lambda y}{y} dy = \frac{f(0^+) + f(0^-)}{2}.
\]

Finally, we have from Riemann-Lebesgue lemma
\[
\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{0}^{\pi} f(-y) \left(\frac{\sin \lambda y}{y} - \frac{\sin \lambda y}{2 \sin \frac{\pi}{2}}\right) dy = 0.
\]

Remark 5. If $f \in BV(S^1) \cap C(S^1)$, then $s_N(f, x)$ converges to $f$ uniformly.

Now we consider $f \in L^p[-\pi, \pi], p \geq 1$. When $p = 2$, we have the well-known result

Theorem 11. Let $f \in L^2[-\pi, \pi]$, then $s_N(f) \to f$ in $L^2[-\pi, \pi]$ as $N \to \infty$. Moreover,
\[
\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |a_n|^2.
\]
On the other hand, we can’t expect \( s_N(f) \to f \) in \( L^1[-\pi, \pi] \) as \( N \to \infty \) if \( f \in L^1[-\pi, \pi] \). We define
\[
T_N : f \mapsto s_N(f).
\]
If for any \( f \in L^1[-\pi, \pi] \), \( T_N f \to f \) in \( L^1[-\pi, \pi] \) as \( N \to \infty \), then uniform boundedness principle implies the existence of \( C > 0 \) such that
\[
\|T_N f\|_{L^1} \leq C \|f\|_{L^1}
\]
holds for any \( f \in L^1[-\pi, \pi] \) and for any \( N \). By duality, we have
\[
\|T_N f\|_{L^\infty} \leq C \|f\|_{L^\infty}
\]
holds for any \( f \in L^\infty[-\pi, \pi] \) and for any \( N \) which is impossible since
\[
\lim_{N \to \infty} \int_{-\pi}^{\pi} |k_N(x)| \, dx = \infty.
\]

**Theorem 12.** Let \( f \in L^p[-\pi, \pi], 1 < p < \infty \), then \( s_N(f) \to f \) in \( L^p[-\pi, \pi] \) as \( N \to \infty \).

We will prove this result later.

### 3. Fourier Transform

If \( f \in L^1(\mathbb{R}^d; \mathbb{C}) \), its Fourier transform \( \mathcal{F}(f) = \hat{f} \) is defined by
\[
\hat{f}(y) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot y} \, dx.
\]
Since
\[
\|\hat{f}\|_{L^\infty} \leq \frac{1}{(2\pi)^{d/2}} \|f\|_{L^1},
\]
the Fourier transform \( \mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \).

Now let’s list some basic properties of Fourier transform:

**Theorem 13.** (i) **Translation:** Let \( f \in L^1(\mathbb{R}^d; \mathbb{C}) \). If \( g(x) = f(x-a) \) and \( h(x) = e^{iax} f(x) \) for some \( a \in \mathbb{R}^d \), then
\[
\hat{g}(y) = e^{-iay} \hat{f}(y) \quad \text{and} \quad \hat{h}(y) = \hat{f}(y-a).
\]

(ii) **Scaling:** Let \( f \in L^1(\mathbb{R}^d; \mathbb{C}) \) and \( \lambda > 0 \). If \( h(x) = f(x/\lambda) \), then
\[
\hat{h}(y) = \lambda^d \hat{f}(\lambda t).
\]

(iii) **Convolution:** Let \( f, g \in L^1(\mathbb{R}^d; \mathbb{C}) \). Then \( f * g \in L^1(\mathbb{R}^d; \mathbb{C}) \) and
\[
(f * g)' = (2\pi)^{d/2} \hat{f} \cdot \hat{g}.
\]
Proof. (iii)

\[(f * g) \hat{\cdot} (y) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (f * g)(x) e^{-ix \cdot y} dx\]
\[
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x - z) g(z) dz \right) e^{-ix \cdot y} dx \\
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x - z) e^{-i(x - z) \cdot y} dx \right) e^{-iz \cdot y} g(z) dz \\
= (2\pi)^{d/2} \hat{f}(y) \hat{g}(y).\]

□

A function \(f \in C^\infty (\mathbb{R}^d; \mathbb{C})\) is said to be rapidly decreasing if for any \(n \geq 0\),

\[
p_n(f) = \sup_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} \left( 1 + |x|^2 \right)^n |(D^\alpha f)(x)| < \infty.
\]

e.g., \(P \cdot D^\alpha f\) is uniformly bounded for every polynomial \(P\) and for every multi-index \(\alpha\). The Schwartz space \(\mathcal{S} = \mathcal{S}_d\) is defined as the function space of rapidly decreasing functions on \(\mathbb{R}^d\). The countable collection of norms \(p_n\) in (3.2) defines a locally convex topology for \(\mathcal{S}\). The topology of \(\mathcal{S}\) is compatible with the metric

\[
d(f, g) = \max_{n \geq 0} \frac{1}{2^n} \frac{p_n(f - g)}{1 + p_n(f - g)}.
\]

It is easy to check \(f \in \mathcal{S}\) implies \(f \in L^1\). Moreover, if \(f \in \mathcal{S}\), then for every multi-index \(\alpha\), \(D^\alpha f, x^\alpha f \in \mathcal{S}\) and we have the following property of Fourier transform.

**Proposition 2.** If \(f \in \mathcal{S}\) and \(\alpha\) is a multi-index, then

\[
(D^\alpha f) \hat{\cdot} (y) = (iy)^\alpha \hat{f}(y), \quad \left(D^\alpha \hat{f}\right)(y) = ((-ix)^\alpha f) \hat{\cdot} (y).
\]

**Proof.** Direct calculation. □

**Theorem 14.** (a) \(\mathcal{S}\) is a Fréchet space.

(b) If \(P\) is a polynomial, \(g \in \mathcal{S}\) and \(\alpha\) is a multi-index, then each of the three mappings

\[
f \rightarrow Pf, f \rightarrow gf, f \rightarrow D_\alpha f
\]

is a continuous linear mapping of \(\mathcal{S}\) into \(\mathcal{S}\).

(c) \(\mathcal{D}(\mathbb{R}^d)\) is dense in \(\mathcal{S}\) and the identity mapping of \(\mathcal{D}(\mathbb{R}^d)\) into \(\mathcal{S}\) is continuous.

**Proof.** (a) Let \(\{f_k\} \subset \mathcal{S}\) be a Cauchy sequence. Then for each \(n\), \(\{f_k\}\) is a Cauchy sequence w.r.t the norm \(p_n\). Hence, for any multi-indices \(\alpha, \beta\), there exists continuous function \(g_{\alpha\beta}\) such that \(x^\alpha D^\beta f_k \rightarrow g_{\alpha\beta}\) uniformly as \(k \rightarrow \infty\). Moreover,

\[
g_{\alpha\beta}(x) = x^\alpha D^\beta g_{00},
\]

so we have \(f_i \rightarrow g_{00}\) in \(\mathcal{S}\). Hence \(\mathcal{S}\) is complete and it is a Fréchet space.

(b) Since \(\mathcal{S}\) is a Fréchet space, a mapping \(\Lambda : \mathcal{S} \rightarrow \mathcal{S}\) is continuous iff \(f_k \rightarrow 0\) implies \(\Lambda f_k \rightarrow 0\). We also recall that \(f_k \rightarrow 0\) iff for each \(n\), \(p_n(f_k) \rightarrow 0\). The continuity of the three mappings follows easily.
The Fourier transform is a continuous linear mapping of \( \mathcal{D}(\mathbb{R}^d) \) for each \( f \in \mathcal{S} \), we define
\[
f_k(x) = f(x) \varphi\left(\frac{x}{k}\right).
\]
Then for any multi-indices \( \alpha, \beta \),
\[
\max_x |x^{\alpha} D^{\beta} (f - f_k)|
\leq \max_{|x| \geq k} \left| \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} x^{\alpha} (D^{\beta-\gamma} f) (x) D^{\gamma} \left[ 1 - \varphi\left(\frac{x}{k}\right) \right] \right|
\leq \max_{|x| \geq k} \left( \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \|D^{\gamma} \varphi\|_{L^\infty} \left(1 + |x|^2 \right) x^{\alpha} (D^{\beta-\gamma} f) (x) \right) \to 0.
\]
Hence, \( f_k \to f \) in \( \mathcal{S} \). So \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( \mathcal{S} \).

The identity mapping of \( \mathcal{D}(\mathbb{R}^d) \) into \( \mathcal{S} \) is continuous iff for each \( K \) compact, the identity mapping of \( \mathcal{D}_K(\mathbb{R}^d) \) into \( \mathcal{S} \) is continuous. Now for \( K \) compact, since \( \mathcal{D}_K(\mathbb{R}^d) \) is a Fréchet space, the identity mapping of \( \mathcal{D}_K(\mathbb{R}^d) \) into \( \mathcal{S} \) is continuous iff \( f_k \to 0 \) in \( \mathcal{D}_K(\mathbb{R}^d) \) implies \( f_k \to 0 \) in \( \mathcal{S} \). The latter statement is easy to verify.

**Theorem 15.** The Fourier transform is a continuous linear mapping of \( \mathcal{S} \) into \( \mathcal{S} \).

**Proof.** For any \( f \in \mathcal{S} \) and for any multi-index \( \alpha, \beta \),
\[
y^{\alpha} (D^{\beta} f) (y) = y^{\alpha} (\xi y)^{\beta} f (\xi y) = (-i)^{\alpha+\beta} (D^{\alpha} (x^{\beta} f)) (x),
\]
which is uniformly bounded, hence \( \hat{f} \in \mathcal{S} \). Let \( f_k \to 0 \) in \( \mathcal{S} \), then for any multi-index \( \alpha, \beta \),
\[
\left\| y^{\alpha} (D^{\beta} f_k) (y) \right\|_{L^\infty} = \left\| (D^{\alpha} (x^{\beta} f_k)) (x) \right\|_{L^\infty} \leq \frac{1}{(2\pi)^{d/2}} \left\| D^{\alpha} (x^{\beta} f_k) \right\|_{L^1}
\leq \frac{1}{(2\pi)^{d/2}} \left(1 + |x|^2 \right)^{\alpha} \left\| D^{\alpha} (x^{\beta} f_k) \right\|_{L^\infty} \leq \frac{1}{(2\pi)^{d/2}} \left(1 + |x|^2 \right)^{\alpha} \left\| D^{\alpha} (x^{\beta} f_k) \right\|_{L^1} \to 0.
\]
Hence \( \hat{f}_k \to 0 \) in \( \mathcal{S} \). So the Fourier transform is continuous.

**Corollary 2.** If \( f \in L^1(\mathbb{R}^d) \), then \( \hat{f} \in C_0(\mathbb{R}^d) \).

**Proof.** For each \( f \in L^1(\mathbb{R}^d) \), there exists \( f_n \in \mathcal{S} \) such that \( f_n \to f \) in \( L^1(\mathbb{R}^d) \). Now (3.1) implies that \( \hat{f}_n \to \hat{f} \) in \( L^\infty(\mathbb{R}^d) \). Since \( \hat{f}_n \in \mathcal{S} \), we conclude \( \hat{f} \in C_0(\mathbb{R}^d) \).

Before introducing the inverse transform, we need the following lemma on approximate delta function.

**Proposition 3.** Let \( \{k_n\} \) be a sequence of functions in \( L^1(\mathbb{R}^d) \) satisfying
(1) \( k_n(x) \geq 0 \) a.e. \( x \in \mathbb{R}^d \);
(2) \[
\int_{\mathbb{R}^d} k_n(x) \, dx = 1;
\]

(3) For any \( \delta > 0 \),
\[
\lim_{n \to \infty} \int_{|x| > \delta} k_n(x) \, dx = 0.
\]
Then for any \( f \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \),
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(y) \, k_n(x - y) \, dx = f(x).
\]
And for any \( f \in L^p(\mathbb{R}^d), 1 \leq p < \infty \),
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(y) \, k_n(x - y) \, dx \text{ converges to } f \text{ in } L^p(\mathbb{R}^d).
\]
We also recall that
\[
\int_{\mathbb{R}} e^{-|x + it|^2} \, dx = \sqrt{\pi}
\]
holds for any \( t \in \mathbb{R} \).

**Theorem 16.** The Fourier transform \( \mathcal{F} : \mathcal{S} \to \mathcal{S} \) is invertible. And for any \( f \in \mathcal{S} \),
\[
f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} \, dy.
\]

**Proof.** We compute
\[
\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} \, dy
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} e^{-\frac{\varepsilon}{2}|y|^2} \, dy
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z) e^{-i(z \cdot y)} e^{ix \cdot y} e^{-\frac{\varepsilon}{2}|y|^2} \, dy \, dz
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{(2\pi \varepsilon)^{d/2}} \int_{\mathbb{R}^d} f(z) e^{-\frac{|z|^2}{2\varepsilon}} \, dz
\]
\[
= f(x).
\]
Here we used the fact that \( \frac{1}{(2\pi)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}} \) is a family of approximate delta functions.

Hence, we can define inverse Fourier transform \( \mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S} \) s.t. for any \( g \in \mathcal{S} \),
\[
\mathcal{F}^{-1}(g)(x) = \hat{g}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(y) e^{ix \cdot y} \, dy.
\]
Apparently, \( \mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S} \) is also continuous.

**Corollary 3.** If \( f, g \in \mathcal{S} \), then \( f * g \in \mathcal{S} \).

**Proof.** \( f, g \in \mathcal{S} \) imply \( \hat{f}, \hat{g} \in \mathcal{S} \), hence \( \hat{f} \hat{g} \in \mathcal{S} \) and
\[
(f * g) = (2\pi)^{d/2} \hat{f} \cdot \hat{g} \in \mathcal{S}.
\]
Hence,
\[
f * g = \mathcal{F}^{-1}((f * g) \hat{)} \in \mathcal{S}.
\]
Proposition 4. If \( f \in L^1(\mathbb{R}^d) \), then as \( \varepsilon \to 0 \), \( f_\varepsilon \) defined by

\[
f_\varepsilon(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} e^{-\frac{\varepsilon}{2} |y|^2} dy
\]

converges to \( f \) in \( L^1 \). If in addition \( \hat{f} \in L^1(\mathbb{R}^d) \), then

\[
f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} dy \text{ a.e. } x \in \mathbb{R}^d.
\]

Proof. We compute

\[
\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} e^{-\frac{\varepsilon}{2} |y|^2} dy = \frac{1}{(2\pi\varepsilon)^{d/2}} \int_{\mathbb{R}^d} f(z) e^{-\frac{|x-z|^2}{2\varepsilon}} dz
\]

Since \( \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-\frac{|z|^2}{2\varepsilon}} \) is a family of approximate delta functions,

\[
\frac{1}{(2\pi\varepsilon)^{d/2}} \int_{\mathbb{R}^d} f(z) e^{-\frac{|x-z|^2}{2\varepsilon}} dz \to f(x) \text{ in } L^1 \text{ as } \varepsilon \to 0.
\]

If in addition \( \hat{f} \in L^1(\mathbb{R}^d) \), Lebesgue dominated convergence implies,

\[
f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} dy \text{ a.e. } x \in \mathbb{R}^d.
\]

Remark 6. If \( f \in L^1(\mathbb{R}^d) \) and \( \hat{f} \in L^1(\mathbb{R}^d) \), then actually \( f, \hat{f} \in C_0(\mathbb{R}^d) \).

Now we prove the well known Parseval formula

Theorem 17. If \( f, g \in \mathcal{S} \), then

\[
\int_{\mathbb{R}^d} f \overline{g} dx = \int_{\mathbb{R}^d} \hat{f} \overline{\hat{g}} dy.
\]

Using the inner product of \( L^2(\mathbb{R}^d) \), we can write

\[
(f, g) = (\hat{f}, \hat{g}).
\]

In particular, when \( f = g \), we have the Plancherel identity

\[
\|\hat{f}\|_{L^2} = \|f\|_{L^2}.
\]

Proof. Using Fubini theorem, we have

\[
\int_{\mathbb{R}^d} f \overline{g} dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}(y) e^{ix \cdot y} \overline{\hat{g}}(x) dxdy
\]

\[
= \int_{\mathbb{R}^d} \hat{f}(y) \overline{\hat{g}}(y) dy.
\]

Corollary 4. If \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), then \( \hat{f} \in L^2(\mathbb{R}^d) \), and

\[
\|\hat{f}\|_{L^2} = \|f\|_{L^2}.
\]
Proof. Choose, $f_n \in S$, such that $f_n \to f$ in $L^1 (\mathbb{R}^d) \cap L^2 (\mathbb{R}^d)$. Since

$$
\| \hat{f}_n - \hat{f}_m \|_{L^2} = \| f_n - f_m \|_{L^2},
$$

$\{ \hat{f}_n \}$ is a Cauchy sequence in $L^2 (\mathbb{R}^d)$, hence there exists $g \in L^2 (\mathbb{R}^d)$, s.t. $\hat{f}_n \to g$ in $L^2 (\mathbb{R}^d)$. On the other hand, $f_n \to f$ in $L^1 (\mathbb{R}^d)$ implies, $\hat{f}_n \to \hat{f}$ in $L^\infty (\mathbb{R}^d)$. Hence $\hat{f} = g \in L^2 (\mathbb{R}^d)$.

Hence we can extend Fourier transform as a linear isometry from $L^2 (\mathbb{R}^d)$ onto itself. Actually, for any $f \in L^2 (\mathbb{R}^d)$, if $\{f_n\} \subset L^1 (\mathbb{R}^d) \cap L^2 (\mathbb{R}^d)$ satisfies $f_n \to f$ in $L^2 (\mathbb{R}^d)$, then

$$
\hat{f} = \lim_{n \to \infty} \hat{f}_n \text{ in } L^2\text{-sense}.
$$

Theorem 18. Let $f \in L^2 (\mathbb{R}^d)$. Define

$$
g_n = \frac{1}{(2\pi)^{d/2}} \int_{|x|<n} f(x) e^{-ix \cdot y} dx,
$$

then

$$
\hat{f} = \lim_{n \to \infty} g_n \text{ in } L^2\text{-sense}.
$$

Moreover, if

$$
h_n = \frac{1}{(2\pi)^{d/2}} \int_{|x|<n} \hat{f}(x) e^{ix \cdot y} dx,
$$

then

$$
f = \lim_{n \to \infty} h_n \text{ in } L^2\text{-sense}.
$$

Proof. We leave the proof as an exercise.

Remark 7. If $f \in L^p (\mathbb{R}^d)$, $1 < p < 2$, then we can write $f = f_1 + f_2$ where $f_1 \in L^1 (\mathbb{R}^d)$ and $f_2 \in L^2 (\mathbb{R}^d)$, and hence we can define $\hat{f} = \hat{f}_1 + \hat{f}_2$.

To define Fourier transform on more general function, we need to introduce tempered distributions.

Definition 4. A continuous linear functional on $S$ is called a tempered distribution. And we use $S'$ to denote the collection of all tempered distributions.

Since $\mathcal{D} (\mathbb{R}^d)$ is dense in $S$ and the identity mapping $I$ of $\mathcal{D} (\mathbb{R}^d)$ into $S$ is continuous, every $\Lambda \in S'$ gives rise to $\Lambda \circ I \in \mathcal{D}' (\mathbb{R}^d)$. Moreover, the map $\Lambda \to \Lambda \circ I$ is one-to-one. Hence, we can view tempered distributions as distributions which have continuous extension to $S$.

Example 4. Every distribution with compact support is tempered.

Example 5. Suppose $\mu$ is a positive Borel measure on $\mathbb{R}^d$ such that

$$
(1 + |x|^2)^{-k} d\mu (x) < \infty.
$$

Then $\mu$ is a tempered distribution.
Example 6. Suppose $1 \leq p < \infty$, $N > 0$ and $g$ is a measurable function on $\mathbb{R}^d$ such that
\[
\int_{\mathbb{R}^d} \left(1 + |x|^2\right)^{-k} |g(x)|^p \, dx < \infty.
\]
Then $g$ is a tempered distribution. In particular, every $L^p$ function is a tempered distribution and so is every polynomial.

Example 7. $e^{\|x\|^2}$ is not a tempered distribution.
Proof. Let $\phi \in \mathcal{D} (\mathbb{R}^d)$ be a radial symmetry function such that $\phi (r) > 0$ when $1 < r < 2$ and $\phi (r) = 0$ otherwise. One can check that $e^{-|x|^2/2} \phi \left(\frac{|x|}{n}\right) \to 0$ in $\mathcal{S}$ while
\[
\int_{\mathbb{R}^d} e^{\|x\|^2} e^{-|x|^2/2} \phi \left(\frac{|x|}{n}\right) \, dx = \infty.
\]

Theorem 19. If $\alpha$ is a multi-index, $P$ is a polynomial, $g \in \mathcal{S}$ and $\Lambda$ is a tempered distribution, then so are $D^\alpha \Lambda$, $P \Lambda$ and $g \Lambda$.
Proof. For any $f \in \mathcal{S}$,
\[
(D^\alpha \Lambda) (f) = (-1)^{|\alpha|} \Lambda (D^\alpha f),
\]
since $D^\alpha : \mathcal{S} \to \mathcal{S}$ is continuous, $D^\alpha \Lambda$ is continuous. Continuity of $P \Lambda$ and $g \Lambda$ follows similarly.

Now we can define Fourier transform of tempered distributions.

Definition 5. For any $u \in \mathcal{S}'$, we define for any $f \in \mathcal{S}$,
\[
\hat{u} (f) = u \left(\hat{f}\right).
\]
Then $\hat{u} \in \mathcal{S}'$.

If $f \in L^1$, then $f$ is a tempered distribution which we denote by $uf$. And for any $g \in \mathcal{S}$, we have
\[
\hat{u}_f (g) = uf \left(\hat{g}\right) = \int_{\mathbb{R}^d} f (x) \hat{g} (x) \, dx \quad = \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f (x) e^{-ix \cdot y} g (y) \, dy \, dx = \int_{\mathbb{R}^d} \hat{f} (y) g (y) = uf (g).
\]

Hence, the definition of Fourier transform of tempered distributions is consistent with the Fourier transform of integrable functions. Similarly, one can show that it is also consistent with the Fourier transform of functions in $L^2 (\mathbb{R}^d)$.

Theorem 20. The Fourier transform is a continuous, linear bijection from $\mathcal{S}'$ onto $\mathcal{S}'$. Its inverse is also continuous.
Proof. Let $W$ be a nbhd of 0 in $\mathcal{S}'$. Then there exists functions $f_1, \ldots, f_n \in \mathcal{S}$ s.t. $\{u \in \mathcal{S}' : |u (f_k)| < 1, 1 \leq k \leq n\} \subset W$. Define,
\[
V = \left\{ u \in \mathcal{S}' : |u (\hat{f}_k)| < 1, 1 \leq k \leq n \right\}.
\]
Then $V$ is a nbhd of 0 in $\mathcal{S}'$, and $\mathcal{F} (V) \subset W$. Hence $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is continuous. Since $\mathcal{F}^4 = I$, $\mathcal{F}$ is a bijection and $\mathcal{F}^{-1} = \mathcal{F}^3$ is continuous.
Theorem 21. If \( u \in S' \), and \( \alpha \) is a multi-index, then
\[
(D^\alpha u)^\wedge = (iy)^\alpha \hat{u},
\]
\[
D^\alpha \hat{u} = ((-ix)^\alpha u)^\wedge.
\]

Proof. By definition.
\( \square \)

Example 8. \( \hat{1} = (2\pi)^{\frac{d}{2}} \delta \) and \( \hat{\delta} = \frac{1}{(2\pi)^{\frac{d}{2}}} \).

Proof. For any \( f \in S \),
\[
\hat{1} (f) = 1 \left( \hat{f} \right) = \int_{\mathbb{R}^d} \hat{f} (y) \, dy = (2\pi)^{\frac{d}{2}} f (0).
\]
An for any \( f \in S \),
\[
\hat{\delta} (f) = \delta \left( \hat{f} \right) = \hat{f} (0) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f (x) \, dx.
\]
\( \square \)

Example 9. Let \( \alpha \) be a multi-index, then
\[
(x^\alpha)^\wedge = (2\pi)^{\frac{d}{2}} i^\alpha D^\alpha \delta.
\]
Since any distribution supported at 0 is a linear combination of distributions of the form \( D^\alpha \delta \), the Fourier transform of \( f \) is supported at 0 iff \( f \) is a nonzero polynomial.

Definition 6. Let \( u \in S' \), and \( \varphi \in S \). We define for any \( x \in \mathbb{R}^d \)
\[
(u \ast \varphi) (x) = u (\varphi (x - \cdot)).
\]

Theorem 22. Let \( u \in S' \), and \( \varphi \in S \). Then \( u \ast \varphi \in C^\infty (\mathbb{R}^d) \). Moreover, for any multi-index \( \alpha \),
\[
D^\alpha (u \ast \varphi) = D^\alpha u \ast \varphi = u \ast D^\alpha \varphi.
\]

Proof. Since the mapping \( x \mapsto \varphi (x - \cdot) \) is continuous from \( \mathbb{R}^d \) into \( S \), \( u \ast \varphi \) is continuous. Now
\[
\frac{(u \ast \varphi) (x + te_i) - (u \ast \varphi) (x)}{t} = u \left( \frac{\varphi (x + te_i - \cdot) - \varphi (x - \cdot)}{t} \right),
\]
since
\[
\frac{\varphi (x + te_i - y) - \varphi (x - y)}{t} \to \partial_i \varphi (x - y) \quad \text{in} \ S,
\]
we have
\[
\partial_i (u \ast \varphi) (x) = \lim_{t \to 0} \frac{(u \ast \varphi) (x + te_i) - (u \ast \varphi) (x)}{t} = u \left( \partial_i \varphi (x - \cdot) \right),
\]
hence \( u \ast \varphi \) has a continuous \( i \)-th partial derivative. By induction, \( u \ast \varphi \in C^\infty (\mathbb{R}^d) \).
\( \square \)

Lemma 1. Let \( u \in S' \), and \( \varphi, \phi \in \mathcal{D} (\mathbb{R}^d) \), we have
\[
(u \ast \varphi) \ast \phi = u \ast (\varphi \ast \phi).
\]
Proof. For any \( h > 0 \), we define the Riemann sum

\[
f_h (x) = h^d \sum_k \varphi (x - kh) \phi (kh)
\]

where for each \( h \) and \( x \), the summation has finitely many nonzero terms. It is straight forward to verify that \( f_h \to \varphi \ast \phi \) in \( \mathcal{S} \). Hence,

\[
u \ast (\varphi \ast \phi) = \lim_{h \to 0^+} u \ast f_h = \lim_{h \to 0^+} h^d \sum_k (u \ast \varphi) (x - kh) \phi (kh) = (u \ast \varphi) \ast \phi.
\]

Here the last equality holds because \((u \ast \varphi) (x - y) \phi (y)\) is Riemann integrable in \( y \).

Now let \( \varphi \in \mathcal{D} (\mathbb{R}^d) \) be the standard mollifier and for any \( \varepsilon > 0 \), we define

\[
\varphi_\varepsilon (x) = \varepsilon^{-n} \varphi \left( \frac{x}{\varepsilon} \right).
\]

Lemma 2. For any \( \phi \in \mathcal{S} \), we have

\[
\phi \ast \varphi_\varepsilon \to \phi \text{ in } \mathcal{S}.
\]

Proof. Direct calculation.

Theorem 23. If \( u \in \mathcal{S}' \), then \( u \ast \varphi_\varepsilon \to u \) in the weak-* topology of \( \mathcal{S}' \).

Proof. For any \( \phi \in \mathcal{S} \), we have

\[
u (\phi) = \left( u \ast \tilde{\phi} \right) (0)
\]

where \( \tilde{\phi} (x) = \phi (-x) \). For any \( \phi \in \mathcal{D} (\mathbb{R}^d) \),

\[
(u \ast \varphi_\varepsilon) (\phi) = \left( (u \ast \varphi_\varepsilon) \ast \tilde{\phi} \right) (0) = \left( u \ast \left( \varphi_\varepsilon \ast \tilde{\phi} \right) \right) (0) = u \left( \varphi_\varepsilon \ast \tilde{\phi} \right).
\]

Since the expression \( u \left( \varphi_\varepsilon \ast \tilde{\phi} \right) \) is continuous for \( \phi \in \mathcal{S} \), we conclude \( u \ast \varphi_\varepsilon \in \mathcal{S}' \) and the above formula holds for \( \phi \in \mathcal{S} \). Since \( \varphi_\varepsilon \ast \tilde{\phi} \to \tilde{\phi} \) in \( \mathcal{S} \), we have

\[
\lim_{\varepsilon \to 0} \left( u \ast \varphi_\varepsilon \right) (\phi) = u (\phi).
\]

Remark 8. Similar proof will yield that if \( u \in \mathcal{D}' (\mathbb{R}^d) \), then \( u \ast \varphi_\varepsilon \to u \) in the weak-* topology of \( \mathcal{D}' (\mathbb{R}^d) \).

Remark 9. The formulas for Fourier transforms of regular functions continue to hold for tempered distributions as long as they are well defined. For example, if \( u \in \mathcal{S}' \) and \( \varphi \in \mathcal{S} \), then

\[
(u \ast \varphi)^\prime = (2\pi)^{\frac{d}{2}} \hat{\varphi} \hat{u}.
\]
Let the concept of distribution function. For any $\phi \in C_c(\mathbb{R}^n)$, let $\xi \in \mathbb{R}^n$ be a nonnegative integer and $1 \leq p \leq \infty$. Sobolev space $W^{k,p}(\Omega)$ is the collection of functions $f \in L^p$ and the distributional derivatives $D^\alpha f \in L^p$ for any $|\alpha| \leq k$. $W^{k,p}(\Omega)$ is a Banach space with norm
\[
\|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,
\]
\[
\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}.
\]
When $p = 2$, $W^{k,2}(\Omega) = H^k(\Omega)$ is a Hilbert space.

Let $f \in H^k(\mathbb{R}^d)$. Then we have for any $|\alpha| \leq k,$
\[
\hat{D^\alpha f} = (iy)^\alpha \hat{f} \in L^2(\mathbb{R}^d).
\]
Hence, $f \in H^k(\mathbb{R}^d)$ if
\[
(1 + |y|^2)^{k/2} \hat{f} \in L^2(\mathbb{R}^d).
\]
Moreover, the norm
\[
\|f\|_{\dot{H}^k} = \left\| (1 + |y|^2)^{k/2} \hat{f} \right\|_{L^2}
\]
is equivalent to the original norm. This expression also allows us to extend the definition of $H^k(\mathbb{R}^d)$ to $k \in \mathbb{R}$. One can identify the dual space of $H^k(\mathbb{R}^d)$ as $H^{-k}(\mathbb{R}^d)$.

**Theorem 24** (Sobolev). Let $\Omega \subset \mathbb{R}^d$ be open and $k > \frac{d}{2}$. Any $f \in H^k(\mathbb{R}^d)$ can be represented by a continuous function.

**Proof.** For any $x \in \Omega$, let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ has compact support in $\Omega$ and $\varphi = 1$ in $B_r(x)$ for some $r > 0$. Then $\varphi f \in H^k(\mathbb{R}^d)$ and $\varphi f = f$ in $B_r(x)$. Since continuity is a local property, we assume $f$ itself is in $H^k(\mathbb{R}^d)$. Now
\[
\int_{\mathbb{R}^d} |\hat{f}|^2 \leq \left\| (1 + |y|^2)^{k/2} \hat{f} \right\|_{L^2} \left\| (1 + |y|^2)^{-k/2} \right\|_{L^2} < \infty.
\]
So we have $\hat{f} \in L^1(\mathbb{R}^d)$ and $f = \left( \hat{f} \right)^\vee \in C_0(\mathbb{R}^d)$. \hfill $\square$

### 4. The Maximal Function

#### 4.1. Distribution Function.

We first recall the concept of distribution function.

**Definition 8.** Let $g$ be defined on $\mathbb{R}^n$. For each $t \geq 0$, we define
\[
\lambda(t) = |\{x \in \mathbb{R}^n : g(x) > t\}|.
\]
The function $\lambda$ is called the distribution function of $|g|$.

The function $\lambda$ is monotone decreasing on $[0, \infty)$ and hence measurable. Any quantity dealing solely with the size of $g$ can be expressed in terms of the distribution function $\lambda(\alpha)$. If $g \in L^p(\mathbb{R}^n)$, then
\[
\int_{\mathbb{R}^n} |g(y)|^p = -\int_0^\infty t^{p-1} d\lambda(t).
\]
And if $g \in L^\infty(\mathbb{R}^n)$, then
\[
\|g\|_\infty = \inf \{t : \lambda(t) = 0\}.
\]
Proposition 5. If \( g \in L^p(\mathbb{R}^n) \), \( p \geq 1 \), then for any \( t > 0 \),
\[
\lambda(t) \leq \frac{1}{t^p} \int_{\mathbb{R}^n} |g(y)|^p \, dy.
\]

Proof.
\[
\int_{\mathbb{R}^n} |g(y)|^p \, dy \geq t^p \int_{|g(x)| > t} dy = t^p \lambda(t).
\]

\( \square \)

4.2. A covering lemma.

Lemma 3. Let \( E \) be a measurable subset of \( \mathbb{R}^n \) which is covered by the union of a family of balls \( \{B_\alpha\}_{\alpha \in \Lambda} \) of bounded radius. Then from this family, we can select a disjoint sequence, \( B_1, B_2, \ldots \), s.t.
\[
\sum |B_\alpha| \geq 5^{-n} |E|.
\]

Proof. We choose \( B_1 \) s.t.
\[
r(B_1) \geq \frac{1}{2} \sup_{\alpha \in \Lambda} r(B_\alpha).
\]

When \( B_1, B_2, \ldots, B_k \) are chosen, we choose \( B_{k+1} \) s.t.,
\[
r(B_{k+1}) \geq \frac{1}{2} \sup \{r(B_\alpha) : \alpha \in \Lambda, B_\alpha \cap B_i = \emptyset, 1 \leq i \leq k\}.
\]

This procedure terminates if the set in the above expression is empty.

If \( \sum |B_\alpha| = \infty \), we are done. So we now assume \( \sum |B_\alpha| < \infty \). Since \( r(B_k) \geq \frac{1}{2} r(B_j) \) for any \( j \geq k \), either \( \{B_k\} \) is finite or \( \lim_{k \to \infty} r(B_k) = 0 \). where \( B_k^* \) is the ball with the same center as \( B_k \) and with radius \( 5r(B_k) \). Given any \( \alpha \in \Lambda \) such that \( B_\alpha \) is not one of \( \{B_k\} \). If \( \{B_k\} \) is finite, then for some \( j \), \( B_j \cap B_\alpha \neq \emptyset \) and \( r(B_j) \geq \frac{1}{2} r(B_\alpha) \) which implies \( B_\alpha \subset B_j^* \). If \( \{B_k\} \) is not finite, then there exists first \( k \), s.t. \( r(B_{k+1}) < \frac{1}{2} r(B_\alpha) \). Hence, for some \( 1 \leq j \leq k \), \( B_j \cap B_\alpha \neq \emptyset \) and \( r(B_j) \geq \frac{1}{2} r(B_\alpha) \) which implies \( B_\alpha \subset B_j^* \). Hence, in any case, \( B_\alpha \subset \bigcup_k B_k^* \). So we have
\[
E \subset \bigcup_{\alpha \in \Lambda} B_\alpha \subset \bigcup_k B_k^*
\]

and
\[
|E| \leq \sum_{\alpha \in \Lambda} |B_k^*| = 5^n \sum |B_k|.
\]

\( \square \)

4.3. Hardy-Littlewood maximal function. We recall that if \( f \in L^1_{loc}(\mathbb{R}^n) \), then
\[
f(x) = \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy
\]
holds for a.e. \( x \in \mathbb{R}^n \). To study such limit, we introduce:

Definition 9. Let \( f \in L^1_{loc}(\mathbb{R}^n) \). The maximal function of \( f \) is defined by
\[
M(f)(x) = \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.
\]
Theorem 25. Let $f$ be a given function defined on $\mathbb{R}^n$
(a) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $Mf$ is finite a.e..
(b) If $f \in L^1(\mathbb{R}^n)$, then for any $t > 0$,
$$m \{ x : (Mf)(x) > t \} \leq \frac{A}{t} \int_{\mathbb{R}^n} |f| \, dx,$$
where we can take $A = 5^n$.
(c) If $f \in L^p(\mathbb{R}^n)$, with $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R}^n)$ and
$$\|Mf\|_p \leq A_p \|f\|_p$$
where
$$A_p = 2 \left( \frac{5^n p}{p - 1} \right)^{\frac{1}{p}}, 1 < p < \infty \text{ and } A_\infty = 1.$$

Proof. (b) Let $E_t = \{ x : (Mf)(x) > t \}$. For any $x \in E_t$, there exits a ball $B_x$ centered at $x$, s.t.,
$$\int_{B_x} |f(y)| \, dy > t |B_x|.$$
The family $\{B_x\}$ is a cover of set $E_t$, hence the covering lemma implies the existence of disjoint sub-collection $\{B_{x_k}\}$ s.t.
$$\sum |B_{x_k}| \geq 5^{-n} |E_t|.$$
Hence,
$$|E_t| \leq 5^n \sum |B_{x_k}| < \frac{5^n}{t} \sum \int_{B_{x_k}} |f| \, dx \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \, dx.$$
(c) The case $p = \infty$ is trivial with $A_\infty = 1$. Now we assume $1 < p < \infty$. For any $t > 0$, we define $f_1(x) = f(x)$ if $|f(x)| > \frac{t}{2}$, $f_1(x) = 0$ otherwise. Then $|f(x)| \leq |f_1(x)| + \frac{t}{2}$ which yields
$$Mf(x) \leq Mf_1(x) + \frac{t}{2}.$$
Hence,
$$E_t = \{ x : (Mf)(x) > t \} \subset \left\{ x : (Mf_1)(x) > \frac{t}{2} \right\}.$$
So we have
$$\lambda(t) = |E_t| \leq m \left\{ x : (Mf_1)(x) > \frac{t}{2} \right\} \leq \frac{2A}{t} \|f_1\|_{L^1} = \frac{2A}{t} \int_{|f(x)| > \frac{t}{2}} |f|.$$
Hence,

$$\|Mf\|^p_p = - \int_0^\infty t^p d\lambda(t) = p \int_0^\infty t^{p-1} \lambda(t) \, dt$$

$$\leq p \int_0^\infty t^{p-1} \frac{2A}{t} \left( \int_{|f| > \frac{1}{2}} |f| \, dx \right) \, dt$$

$$= 2pA \int_{\mathbb{R}^n} |f(x)| \left( \int_0^{2|f|} t^{p-2} \, dt \right) \, dx$$

$$= \frac{2pA}{p-1} \int_{\mathbb{R}^n} |f(x)| (2|f|)^{p-1} \, dx$$

$$= \frac{2pA}{p-1} \|f\|_p^p = A_p \|f\|_p^p.$$

\[ \square \]

**Remark 10.** $A_p \to \infty$ as $p \to 1$ which suggests that part (c) will not hold for $p = 1$. Actually, $Mf$ is never integrable unless $f = 0$ a.e.. The estimate in (c) is called type $(p, p)$, while the estimate in (b) is called type weak $(1, 1)$.

**Corollary 5.** If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$f(x) = \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy$$

holds for a.e. $x \in \mathbb{R}^n$.

**Proof.** We assume $f \in L^1(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, we define

$$f_r(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy$$

and

$$\Omega f(x) = \left| \limsup_{r \to 0^+} f_r(x) - \limsup_{r \to 0^+} f_r(x) \right|.$$

For any $\varepsilon > 0$, we choose, for any $\delta > 0$, $h \in C(\mathbb{R}^n)$ and denote $g = f - h$ s.t.

$$\|g\|_1 = \|f - h\|_1 < \delta.$$

Then $\Omega f(x) = \Omega g(x) \leq 2Mg$. Now

$$m \{ x : \Omega f(x) > \varepsilon \} = m \{ x : \Omega g(x) > \varepsilon \}$$

$$\leq m \{ x : Mg(x) > \frac{\varepsilon}{2} \} \leq \frac{2 \cdot 5^n}{\varepsilon} \|g\|_1 \leq 2 \cdot 5^n \delta.$$

Since $\delta$ can be arbitrarily small, we have

$$m \{ x : \Omega f(x) > \varepsilon \} = 0.$$

Hence $\Omega f(x) = 0$ a.e. $x \in \mathbb{R}^n$. Hence $\lim_{r \to 0^+} f_r(x)$ exists a.e.. On the other hand, since $f_r \to f$ in $L^1$, there exists a subsequence $r_k$ s.t. $f_{r_k} \to f$ a.e. $x \in \mathbb{R}^n$.

Hence,

$$f(x) = \lim_{r \to 0^+} f_r(x)$$

holds for a.e. $x \in \mathbb{R}^n$. \[ \square \]
4.4. Generalized maximal function.

**Definition 10.** Let $\mathcal{F}$ be a family of measurable subsets of $\mathbb{R}^n$. We say $\mathcal{F}$ is regular, if there exists $c > 0$ s.t. if $S \in \mathcal{F}$, then there exists a ball $B$ centered at 0, s.t. $S \subset B$, $m(S) \geq cm(B)$.

Here are some examples:

**Example 10.** The family $\mathcal{F} = \{\delta U\}_{\delta > 0}$ is regular if $U$ is bounded and measurable with $m(U) > 0$.

**Example 11.** The family $\mathcal{F}$ of all cubes with the property that their distance from the origin is bounded by a constant multiple of their diameter.

**Example 12.** The family $\mathcal{F}$ of all balls containing the origin.

**Example 13.** Any subfamily of a regular family is regular.

Given a regular family $\mathcal{F}$ of measurable subsets of $\mathbb{R}^n$. We define the maximal function

$$M_\mathcal{F}(f)(x) = \sup_{S \in \mathcal{F}} \frac{1}{m(S)} \int_S |f(x - y)| \, dy.$$  

Let $c > 0$ be the number in the Definition of regular family, then we have

$$M_\mathcal{F}(f)(x) \leq \frac{1}{c} M(f)(x).$$

Hence strong $(p, p)$, $1 < p \leq \infty$ and weak $(1, 1)$ estimates hold for $M_\mathcal{F}$.

**Corollary 6.** Suppose $\mathcal{F}$ is a regular family of measurable subsets of $\mathbb{R}^n$ such that

$$\inf_{S \in \mathcal{F}} m(S) = 0.$$  

If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$f(x) = \lim_{S \in \mathcal{F}, m(S) \to 0} \frac{1}{m(S)} \int_S f(x - y) \, dy$$

holds for a.e. $x \in \mathbb{R}^n$.

**Definition 11.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. A point $x \in \mathbb{R}^n$ is said to be a Lebesgue point of $f$ if

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0.$$  

The Lebesgue set of $f$ is the collection of all Lebesgue points of $f$.

**Theorem 26.** Let $f \in L^1_{loc}(\mathbb{R}^n)$.

(a) The complement of Lebesgue set is of measure zero.

(b) Suppose $\mathcal{F}$ is a regular family of measurable subsets of $\mathbb{R}^n$ such that

$$\inf_{S \in \mathcal{F}} m(S) = 0.$$  

Then

$$f(x) = \lim_{S \in \mathcal{F}, m(S) \to 0} \frac{1}{m(S)} \int_S f(x - y) \, dy$$

holds for every Lebesgue point $x \in \mathbb{R}^n$. 
Proof. (a) For any $s \in \mathbb{Q}$,

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - s| \, dy = |f(x) - s|$$

holds for a.e. $x \in \mathbb{R}^n$. We define the exceptional set $E_s$, then $|E_s| = 0$. Let $E = \bigcup_{s \in \mathbb{Q}} E_s$, then $|E| = 0$. For any $x \notin E$, we have

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - s| \, dy = |f(x) - s|$$

holds for every $s \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $x$ is a Lebesgue point.

(b) Let $x$ be a Lebesgue point. Then

$$\left| \frac{1}{m(S)} \int_S f(x-y) \, dy - f(x) \right| \leq \frac{1}{m(S)} \int_S |f(x-y) - f(x)| \, dy \leq \frac{1}{c|B|} \int_B |f(x-y) - f(x)| \, dy \to 0$$

as $|B| \to 0$. \hfill \Box

4.5. An interpolation theorem for $L^p$. Let $1 \leq p, q \leq \infty$ and $T$ be a mapping from $L^p(\mathbb{R}^n)$ to measurable functions in $\mathbb{R}^n$. Then $T$ is of type $(p, q)$ if for some $A > 0$,

$$\|Tf\|_q \leq A \|f\|_p \text{ holds for every } f \in L^p(\mathbb{R}^n).$$

Similarly, if in addition, $q < \infty$, $T$ is of weak type $(p, q)$ if for some $A > 0$,

$$m \{ x : |Tf(x)| > t \} \leq \left( \frac{A \|f\|_p}{t} \right)^q \text{ holds for every } f \in L^p(\mathbb{R}^n).$$

When $q = \infty$, $T$ is of weak type $(p, q)$ if it is of type $(p, q)$.

If $T$ is of type $(p, q)$, then $T$ is of weak type $(p, q)$. Assume $q < \infty$ and $T$ is of type $(p, q)$. For every $f \in L^p(\mathbb{R}^n)$,

$$m \{ x : |Tf(x)| > t \} \leq \frac{1}{t^q} \|Tf\|_q^q \leq \left( \frac{A \|f\|_p}{t} \right)^q.$$

Hence, $T$ is of weak type $(p, q)$.

The following is a special case of Marcinkiewicz interpolation theorem:

**Theorem 27** (Marcinkiewicz). Suppose $1 < r \leq \infty$ and $T$ maps $L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ into the space of measurable functions in $\mathbb{R}^n$. Suppose that $T$ is

(i) subadditive: For any $f, g \in L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$,

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|;$$

(ii) of weak type $(1, 1)$: For any $f \in L^1(\mathbb{R}^n)$,

$$m \{ x : |Tf(x)| > t \} \leq \frac{A_1}{t} \|f\|_1.$$
(iii) of weak type \((r, r)\): If \(r < \infty\), for any \(f \in L^r(\mathbb{R}^n)\),
\[
m \{ x : |Tf(x)| > t \} \leq \left( \frac{A_r \|f\|_r}{t} \right)^r
\]
and if \(r = \infty\),
\[
\|Tf\|_\infty \leq A_\infty \|f\|_\infty.
\]
Then for any \(1 < p < r\),
\[
\|Tf\|_p \leq A_p \|f\|_p
\]
holds for every \(f \in L^p(\mathbb{R}^n)\).

where
\[
A_p = \left[ \left( \frac{2^r A_1}{p-1} + \frac{(2A_r)^r}{r-p} \right) p \right]^{\frac{1}{p}} \text{ if } r < \infty.
\]

Proof. The case \(r = \infty\) follows from the proof of \((p, p)\) estimate for maximal function. Now we assume \(1 < r < \infty\). Fix \(f \in L^p\). For any \(t > 0\), we define
\[
f_1 = f \chi_{|f| > t} \text{ and } f_2 = f \chi_{|f| \leq t}.
\]
Then \(f_1 \in L^1\) and \(f_2 \in L^r\). Since
\[
|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|,
\]
we have
\[
\{ x : |Tf(x)| > t \} \subset \left\{ x : |Tf_1(x)| > \frac{t}{2} \right\} \cup \left\{ x : |Tf_2(x)| > \frac{t}{2} \right\}.
\]
Hence,
\[
\lambda(t) = |\{ x : |Tf(x)| > t \}|
\leq \left| \left\{ x : |Tf_1(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x : |Tf_2(x)| > \frac{t}{2} \right\} \right|
\leq \frac{2A_1}{t} \|f_1\|_1 + \left( \frac{2A_r \|f_2\|_r}{t} \right)^r
\]
\[
= \frac{2A_1}{t} \int_{|f| > t} |f| \, dx + \left( \frac{2A_r}{t} \right)^r \int_{|f| \leq t} |f|^r \, dx.
\]
Recall that
\[
\|Tf\|_p^p = p \int_0^\infty t^{p-1} \lambda(t) \, dt.
\]
Now we have
\[
\int_0^\infty t^{p-1} \left( \frac{1}{t} \int_{|f| > t} |f| \, dx \right) \, dt = \frac{1}{p-1} \|f\|_p^p
\]
and
\[
\int_0^\infty t^{r-1} \left( \frac{1}{t^r} \int_{|f| \leq t} |f|^r \, dx \right) \, dt = \frac{1}{r-p} \|f\|_p^p.
\]
Hence,
\[
\|Tf\|_p^p \leq \left( \frac{2A_1}{p-1} + \frac{(2A_r)^r}{r-p} \right) p \|f\|_p^p = A_p \|f\|_p^p.
\]
\qed
4.6. **Behavior near general points of measurable sets.** Suppose $E$ is a given measurable subset of $\mathbb{R}^n$. We say $x \in \mathbb{R}^n$ is a point of density of $E$ if
\[
\lim_{r \to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = 1.
\]

**Proposition 6.** Almost every point $x \in E$ is a point of density of $E$, and almost every point $x \notin E$ is not a point of density of $E$.

Now let $E$ be a closed set. For any $x \in \mathbb{R}^n$, we denote $\delta(x) = \text{dist}(x, E)$.

**Proposition 7.** Let $E$ be a closed set. Then for almost every $x \in E$, $\delta(x+y) = o(|y|)$. This holds in particular if $x$ is a point of density of $E$.

**Proof.** Let $x$ be a point of density of $F$ and suppose $\varepsilon > 0$ is given. If $\delta(x+y) > \varepsilon |y|$, then
\[
\frac{|E \cap B(1+\varepsilon)|y|(x)}{|B(1+\varepsilon)|y|(x)|} \leq 1 - \left(\frac{\varepsilon}{1 + \varepsilon}\right)^n
\]
which is impossible if $|y|$ is sufficiently small. \qed

Now we consider the integral of marcinkiewicz:
\[
I(x) = \int_{|y| \leq 1} \frac{\delta(x+y)}{|y|^{n+1}} dy.
\]

We need the following lemma:

**Lemma 4.** Let $E$ be a closed set such that $|E^c| < \infty$ and
\[
I_*(x) = \int_{\mathbb{R}^n} \frac{\delta(x+y)}{|y|^{n+1}} dy.
\]
Then $I_*(x) < \infty$ for a.e. $x \in E$. Moreover,
\[
\int_E I_*(x) dx \leq n\omega_n |E^c|
\]
where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

**Proof.**
\[
\int_E I_*(x) dx = \int_E \int_{\mathbb{R}^n} \frac{\delta(x+y)}{|y|^{n+1}} dy dx
\]
\[
= \int_E \int_{E^c} \frac{\delta(y)}{|x-y|^{n+1}} dy dx
\]
\[
= \int_{E^c} \delta(y) \left( \int_E \frac{1}{|x-y|^{n+1}} dx \right) dy
\]
\[
\leq \int_{E^c} \delta(y) \left( \int_{|x| \geq \delta(y)} \frac{1}{|x|^{n+1}} dx \right) dy
\]
\[
= n\omega_n \int_{E^c} dy = n\omega_n |E^c|.
\]
\qed
Theorem 28. Let $E$ be a closed set.
(a) When $x \notin E$, $I(x) = \infty$;
(b) For almost every $x \in F$, $I(x) < \infty$.

Remark 11. We can also define for $\lambda > 0$,
$$f^{(\lambda)}(x) = \int_{|y| \leq 1} \delta^\lambda(x + y) \, dy.$$  

4.7. Decomposition in cubes of open sets in $\mathbb{R}^n$.

Theorem 29 (Calderón Zygmund decomposition). Let $f$ be a nonnegative integrable function on $\mathbb{R}^n$ and $\alpha > 0$. Then there exists a decomposition of $\mathbb{R}^n$ so that
(i) $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$;
(ii) $f(x) \leq \alpha$ a.e. on $F$.
(iii) $\Omega$ is the union of closed cubes, $\Omega = \bigcup_k Q_k$, whose interiors are disjoint and for each $k$,
$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} f(x) \, dx \leq 2^n \alpha.$$
Moreover,
$$|\Omega| \leq \frac{1}{\alpha} \|f\|_1.$$

Proof. We decompose $\mathbb{R}^n$ into a mesh of equal cubes such that
$$\frac{1}{|Q|} \int_Q f(x) \, dx \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} f(x) \, dx \leq \alpha.$$  

Let $Q'$ be a fixed cube, we divide it into $2^n$ congruent cubes. For each new cube $Q''$,

Case 1:
$$\frac{1}{|Q''|} \int_{Q''} f(x) \, dx \leq \alpha.$$  

Then we continue the process.

Case 2:
$$\frac{1}{|Q''|} \int_{Q''} f(x) \, dx > \alpha.$$  

We select $Q''$ as one of $Q_k$.

Let $\Omega = \bigcup_k Q_k$, and $F = \mathbb{R}^n \setminus \Omega$. Then $f(x) \leq \alpha$ a.e. on $F$.

Theorem 30. Let $F$ be a nonempty closed set in $\mathbb{R}^n$. Then its complement $\Omega = F^c$ can be written as a union of cubes
$$\Omega = \bigcup_{k=1}^{\infty} Q_k,$$
whose interiors are disjoint and for each $k$,
$$\text{diam } Q_k \leq \text{dist } (Q_k, F) \leq 4 \text{ diam } Q_k.$$  

Proof. Consider a mesh $M_0$ of cubes with unit length whose vertices are integral points. Let $M_k = 2^{-k} M_0$. We also define for some $c > 0$,
$$\Omega_k = \{ x : c 2^{-k} < \text{dist } (x, F) < c 2^{-(k+1)} \}.$$
Let
\[ F_0 = \bigcup_k \{ Q \in M_k : Q \cap \Omega_k \neq \emptyset \} . \]

Then with \( c = 2\sqrt{n} \), we have

\[ \text{diam } Q \leq \text{dist } (Q, F) \leq 4 \text{diam } Q \]

for each \( Q \in F_0 \). We have \( \Omega = \bigcup_{Q \in F_0} Q \). We observe that if \( Q_0 \cap Q_1 \neq \emptyset \), then one cube contains the other. For each \( Q \in F_0 \), let \( Q' \) be the maximal cube in \( F_0 \) which contain \( Q \). Such maximal cube is unique and all the interiors of maximal cubes are disjoint. Let \( \{ Q_k \} \) be the collection of all maximal cubes in \( \Omega \). We have the desired properties. \( \square \)

With the help of Theorem 30, we can improve certain aspects of the Calderon Zygmund decomposition.

**Theorem 31.** Let \( f \) be a nonnegative integrable function on \( \mathbb{R}^n \) and \( \alpha > 0 \). Let
\[ F = \{ x : Mf(x) \leq t \} \]

and
\[ \Omega = F^c = \{ x : Mf(x) > t \} . \]

We can write \( \Omega \) as the union of closed cubes, \( \Omega = \bigcup_k Q_k \), whose interiors are disjoint and for each \( k \),
\[ \text{diam } Q_k \leq \text{dist } (Q_k, F) \leq 4 \text{diam } Q_k . \]

Moreover, for each \( k \),
\[ \frac{1}{|Q_k|} \int_{Q_k} f(x) \, dx \leq c \alpha \]

where \( c \) is a number depending only on \( n \).

**Proof.** Since \( F \) is closed, its complement \( \Omega = F^c \) can be written as a union of cubes
\[ \Omega = \bigcup_{k=1}^{\infty} Q_k , \]
whose interiors are disjoint and for each \( k \),
\[ \text{diam } Q_k \leq \text{dist } (Q_k, F) \leq 4 \text{diam } Q_k . \]

For each cube \( Q_k \), let \( p_k \in F \) be such that \( \text{dist } (Q_k, F) = \text{dist } (Q_k, p_k) \). Let
\[ r_k = \max_{x \in Q_k} \text{dist } (x, p_k) \leq 5 \text{diam } Q_k , \]
then \( Q_k \subset B_{r_k} (p_k) \). So we have
\[ t \geq Mf(p_k) \geq \frac{1}{|B_{r_k} (p_k)|} \int_{B_{r_k} (p_k)} |f(x)| \, dx \]
\[ \geq \frac{1}{|B_{r_k} (p_k)|} \int_{Q_k} f(x) \, dx \]
\[ \geq \frac{1}{c |Q_k|} \int_{Q_k} f(x) \, dx . \]
Hence
\[ \frac{1}{|Q_k|} \int_{Q_k} f(x) \, dx \leq ct. \]
\[ \square \]

**Remark 12.** We also have
\[ |\Omega| \leq \frac{2^n}{t} \|f\|_1. \]

5. **Singular Integral**

5.1. **Convolution of measures.** The dual space of \( C_0(\mathbb{R}^n) \) is \( \mathcal{M}(\mathbb{R}^n) \), the space of all finite Borel measures. For any \( \mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^n) \), \( \mu = \mu_1 * \mu_2 \) is defined by
\[ \mu(f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + y) \, d\mu_1(x) \, d\mu_2(y). \]
We have \( \mu_1 * \mu_2 = \mu_2 * \mu_1 \) and \( \|\mu\| \leq \|\mu_1\| \|\mu_2\|. \)
Let \( 1 \leq p < \infty \). If \( f \in L^p(\mathbb{R}^n) \), then \( f * \mu \in L^p(\mathbb{R}^n) \) and
\[ \|f * \mu\|_p \leq \|f\|_p \|\mu\|. \]

The map \( T : f \mapsto f * \mu \) is a bounded linear operator from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) which commutes with translation. Such mappings can be classified.

**Theorem 32.** Suppose \( T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \) is linear, bounded and commutes with translations. Then there exists a unique tempered distribution \( u \in S' \) such that
\[ T(\varphi) = \varphi * u \text{ for any } \varphi \in S. \]
Before presenting its proof, we consider special cases when \( p = q \) where we can say more on the tempered distribution \( u \).

**Theorem 33.** Suppose \( T : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \) is linear, bounded and commutes with translations. Then there exists a unique \( \mu \in \mathcal{M}(\mathbb{R}^n) \) such that \( T(f) = f * \mu \) for any \( f \in L^1(\mathbb{R}^n) \). Moreover, \( \|T\| = \|\mu\|. \)
**Proof.** Let \( u \in S' \) be such that \( T(\varphi) = \varphi * u \) for any \( \varphi \in S \). We have for any \( \varphi \in S \),
\[ \|\varphi * u\|_1 \leq \|T\| \|\varphi\|_1. \]
Suppose \( \varphi_\varepsilon \) is the standard mollifier, we have
\[ \|\varphi_\varepsilon \cdot u\|_1 \leq \|T\|. \]
Hence, \( \varphi_{\varepsilon_k} \cdot u \to \mu \) for a subsequence \( \varepsilon_k \to 0 \). On the other hand, we also have \( \varphi_{\varepsilon_k} \cdot u \to u \) in \( S' \). So we have \( u = \mu \). \( \|T\| = \|\mu\| \) can be verified since
\[ \lim_{\varepsilon \to 0} \|\varphi_\varepsilon \cdot \mu\| = \|\mu\|. \]
\[ \square \]

**Theorem 34.** Suppose \( T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is linear, bounded and commutes with translations. Then there exists a multiplier \( m \in L^{\infty}(\mathbb{R}^n) \), such that
\[ \hat{T}f = m \cdot \hat{f} \]
for any \( f \in L^2(\mathbb{R}^n) \).
Proof. Let \( u \in S' \) be such that \( T(\varphi) = \varphi * u \) for any \( \varphi \in S \). Then for \( \varphi = e^{-\frac{x^2}{2}} \),
\[
\hat{T}\varphi = (2\pi)^{n/2} \hat{\varphi} \cdot \hat{u}.
\]
For any \( \phi \in D(\mathbb{R}^n) \), since \( \hat{\varphi} = e^{-\frac{x^2}{2}} \) we have \( \frac{\partial}{\partial x_i} \) \( \hat{\varphi} \in S \) and
\[
\frac{\partial}{\partial x_i} \hat{T}\varphi = (2\pi)^{n/2} \phi \cdot \hat{u} \in L^2.
\]
So we have \( \hat{u} \in L^2_{\text{loc}}(\mathbb{R}^n) \). Now for any \( \varphi \in S \), we have
\[
(2\pi)^{n/2} \| \hat{\varphi} \cdot \hat{u} \|_2 = \| \hat{T}\varphi \|_2 = \| T\varphi \|_2 \leq A \| \hat{\varphi} \|_2.
\]
Hence \( \hat{u} \in L^\infty(\mathbb{R}^n) \). Let \( m = (2\pi)^{n/2} \hat{u} \in L^\infty(\mathbb{R}^n) \). We have for any \( \varphi \in S \),
\[
\hat{T}\varphi = m\hat{\varphi}.
\]
Since \( S \) is dense in \( L^2(\mathbb{R}^n) \), the continuities of \( T \) and Fourier transform imply
\[
\hat{T}f = m \hat{f}
\]
for any \( f \in L^2(\mathbb{R}^n) \). \( \square \)

Lemma 5. Suppose \( 1 < p < \infty \) and \( T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \) is linear, bounded and commutes with translations. Then \( T \), restricted to \( L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \), extends to a map \( T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \) which is linear, bounded and commutes with translations.

Proof. Let \( u \in S' \) be such that \( T(\varphi) = \varphi * u \) for any \( \varphi \in S \). We observe that for any \( \psi \in C_0^\infty(\mathbb{R}^n) \), if \( u \in C_0^\infty(\mathbb{R}^n) \), we have
\[
\int_{\mathbb{R}^n} (\psi * u)(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(y) \, u(x-y) \varphi(x) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(y) \, u(x) \varphi(x+y) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^n} \psi(y) \, u(x) \varphi(x+y) \, dx \, dy
\]
Using Lemma 1, we can show that for any \( \varphi, \psi \in C_0^\infty(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} (\psi * u)(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \psi(y) \, \varphi(x+y) \, (y) \, dx \, dy,
\]
i.e.,
\[
\int_{\mathbb{R}^n} (T\psi)(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \psi(y) \, (T\varphi)(x+y) \, dx \, dy.
\]
For fixed \( \psi \in C_0^\infty(\mathbb{R}^n) \),
\[
\left| \int_{\mathbb{R}^n} (T\psi)(x) \varphi(x) \, dx \right| \leq \| \psi \|_q \| T\varphi \|_p \leq \| T \| \| \psi \|_q \| \varphi \|_p
\]
holds for any \( \varphi, \psi \in C_0^\infty(\mathbb{R}^n) \), hence
\[
\| T\psi \|_q \leq \| T \| \| \psi \|_q.
\]
So \( T \) can be extended to a continuous linear functional \( T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \). \( \square \)
Theorem 35. Suppose $1 < p < \infty$ and $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is linear, bounded and commutes with translations. Then there exists a multiplier $m \in L^\infty(\mathbb{R}^n)$, such that

$$\hat{T}\varphi = m \cdot \hat{\varphi}$$

for any $\varphi \in \mathcal{S}$.

Proof. $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $T$ can be extended to $T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$ 

Hence Marcinkiewicz interpolation theorem implies $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The conclusion now follows from Theorem 34. □

Now we prepare to prove Theorem 32.

Lemma 6. If $f \in W^{n+1,p}(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $f$ equals a.e. a continuous function $g$ satisfying

$$\|g\|_\infty \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_p.$$ 

Proof. We first assume $f \in \mathcal{D}(\mathbb{R}^n)$. Then we have

$$\left| \hat{f}(x) \right| \leq c \left( 1 + |x|^2 \right)^{-\frac{n+1}{2}} \sum_{|\alpha| \leq n+1} |x|^{|\alpha|} \left| \hat{f}(x) \right|$$

$$= c \left( 1 + |x|^2 \right)^{-\frac{n+1}{2}} \sum_{|\alpha| \leq n+1} \left| \hat{D^\alpha f}(x) \right|$$

$$\leq c \left( 1 + |x|^2 \right)^{-\frac{n+1}{2}} \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_1.$$ 

Hence

$$\|f\|_\infty \leq c_2 \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_1 \int_{\mathbb{R}^n} \left( 1 + |x|^2 \right)^{-\frac{n+1}{2}} \, dx \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_1.$$ 

Hence the case $p = 1$ follows from the density of $\mathcal{D}(\mathbb{R}^n)$ in $W^{n+1,1}(\mathbb{R}^n)$.

For $1 < p \leq \infty$, $f \in W^{n+1,p}(\mathbb{R}^n)$ implies $\varphi f \in W^{n+1,1}(\mathbb{R}^n)$ where $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is a cutoff function s.t. $\varphi = 1$ on $B_1(0)$ and $\varphi = 0$ on $\mathbb{R}^n \setminus B_2(0)$. Then $\varphi f$ is continuous, and

$$|f(0)| = |\varphi f(x)| \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha (\varphi f)\|_1 \leq C_2 \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_{L^1(B_2(x))}$$

$$\leq C_3 \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_p.$$ 

Since the choice of origin is arbitrary, we have

$$\|g\|_\infty \leq C_3 \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_p.$$ 

□

Now we prove Theorem 32:
Proof of Theorem 32. Let \( \varphi \in \mathcal{S} \). Since

\[
\frac{\tau_{te_1} \varphi - \varphi}{t} \rightarrow \partial_{x_1} \varphi \text{ in } L^p \text{ as } t \rightarrow 0^+,
\]

we have

\[
\frac{(T \varphi)(x + te_1) - (T \varphi)(x)}{t} = T \left( \frac{\tau_{te_1} \varphi - \varphi}{t} \right)(x) \rightarrow T(\partial_{x_1} \varphi)
\]

in \( L^q \) as \( t \rightarrow 0^+ \). Hence \( \partial_{x_1} (T \varphi) \) is continuous and

\[
\left| (T \varphi)(0) \right| \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha (T \varphi)\|_q = C \sum_{|\alpha| \leq n+1} \|T (D^\alpha \varphi)\|_q
\]

\[
\leq C \|T\| \sum_{|\alpha| \leq n+1} \|D^\alpha \varphi\|_p.
\]

Hence the map \( \varphi \rightarrow (T \varphi)(0) \) is a continuous map on \( \mathcal{S} \), and there exists a unique \( \tilde{u} \in \mathcal{S}' \) s.t.

\[
(T \varphi)(0) = \tilde{u}(\varphi).
\]

Now we define \( u \in \mathcal{S}' \) s.t \( u(\varphi) = \tilde{u}(\varphi) \), then we have

\[
(T \varphi)(x) = T(\tau_x \varphi)(0) = \tilde{u}(\varphi(x + \cdot)) = u(\varphi(x - \cdot)) = (\varphi * u)(x).
\]

\[
\square
\]

5.2. Singular integrals: the heart of the matter. In this section, we will study a typical singular integral.

**Theorem 36.** Let \( K \in L^2(\mathbb{R}^n) \). We suppose:

(a) \( |\hat{K}(x)| \leq B \).

(b) \( K \in C^1(\mathbb{R}^n \setminus \{0\}) \) and

\[
|\nabla K(x)| \leq B |x|^{n+1}.
\]

For \( f \in L^1 \cap L^p, 1 < p < \infty \), define

\[
(T f)(x) = \int_{\mathbb{R}^n} K(x - y) f(y) \, dy.
\]

Then

\[
\|T f\|_p \leq A_p \|f\|_p
\]

where \( A_p = c(n,p)B \).

**Proof.** We assume \( B = 1 \).

Step 1. \( T \) is of type \((2,2)\). Since \( T f = K * f \),

\[
\hat{T f} = (2\pi)^{-\frac{n}{2}} \hat{K} \cdot \hat{f},
\]

we have

\[
\|T f\|_2 = \|\hat{T f}\|_2 \leq (2\pi)^{-\frac{n}{2}} \|\hat{f}\|_2 = (2\pi)^{-\frac{n}{2}} \|f\|_2.
\]

Step 2. \( T \) is of weak type \((1,1)\). Fix \( \alpha > 0 \), we define

\[
F = \{x : Mf(x) \leq t\}
\]

and

\[
\Omega = F^c = \{x : Mf(x) > t\}.
\]
From Theorem 31, we can write $\Omega$ as the union of closed cubes, $\Omega = \bigcup_k Q_k$, whose interiors are disjoint and for each $k$,

$$\text{diam } Q_k \leq \text{dist } (Q_k, F) \leq 4 \text{ diam } Q_k.$$ Moreover, for each $k$,

$$\frac{1}{|Q_k|} \int_{Q_k} f(x) \, dx \leq c\alpha.$$ Moreover,

$$|\Omega| \leq \frac{c}{\alpha} \|f\|_1.$$ Here we use $c$ to denote different constants depending only on $n$.

Now we define

$$g(x) = \begin{cases} f(x) & \text{for } x \in F, \\
\frac{1}{|Q_k|} \int_{Q_k} f(x) \, dx & \text{for } x \in Q_k^c. \end{cases}$$

Then $g$ is defined a.e. Let $b = f - g$, then we have $b(x) = 0$ for $x \in F$ and

$$\int_{Q_k} b(x) \, dx = 0$$

for each cube $Q_k$.

Since

$$\int_{\mathbb{R}^n} g^2(x) \, dx = \int_F f^2(x) \, dx + \int_{\Omega} g^2(x) \, dx \leq \alpha \int_F |f(x)| \, dx + (c\alpha)^2 |\Omega| \leq c\alpha \|f\|_1.$$ We have

$$\left\{ x : |Tg(x)| > \alpha \right\} \leq \frac{4}{\alpha^2} \|Tg\|_2^2 \leq \frac{c}{\alpha^2} \|g\|_2^2 \leq \frac{c}{\alpha} \|f\|_1.$$ Now we estimate $Tb$. Write

$$b_k(x) = b(x) \chi_{Q_k}.$$ We have for any $x \in F$,

$$|Tb_k(x)| = \left| \int_{Q_k} K(x-y) b(y) \, dy \right| = \left| \int_{Q_k} [K(x-y) - K(x)] b(y) \, dy \right| \leq c \text{ diam } Q_k \int_{Q_k} \frac{|b(y)|}{|x-y|^{n+\tau}} \, dy.$$ Since

$$\int_{Q_k} |b(y)| \, dy \leq \int_{Q_k} |f(y)| \, dy + c\alpha |Q_k| \leq c\alpha |Q_k|,$$ and

$$\text{diam } Q_k \leq \text{dist } (Q_k, F) \leq 4 \text{ diam } Q_k.$$ we have for any $x \in F$,

$$|Tb_k(x)| \leq c\alpha \int_{Q_k} \frac{\delta(y)}{|x-y|^{n+\tau}} \, dy.$$
where \( \delta (y) = \text{dist} (y, F) \). Hence,
\[
|Tb (x)| \leq c \alpha \int_{\Omega} \frac{\delta (y)}{|x-y|^{n+1}} dy.
\]
Applying Lemma 4, we have
\[
\int_{F} |Tb (x)| \, dx \leq c \alpha |\Omega| \leq c \|f\|_{1}.
\]
Hence,
\[
\left| \{ x \in \mathbb{R}^{n} : |Tb (x)| > \frac{\alpha}{2} \} \right| \leq \left| \{ x \in F : |Tb (x)| > \frac{\alpha}{2} \} \right| + |\Omega| \\
\leq \frac{c}{\alpha} \|f\|_{1}.
\]
Hence
\[
|\{ x : |Tf (x)| > \alpha \}| \leq \left| \{ x : |Tg (x)| > \frac{\alpha}{2} \} \right| + \left| \{ x : |Tb (x)| > \frac{\alpha}{2} \} \right| \\
\leq \frac{c}{\alpha} \|f\|_{1}.
\]
Step 3. Interpolation theorem implies the case \( 1 < p < 2 \).
Step 4. The case \( 2 < p < \infty \) comes from the standard duality argument since
\[
\int_{\mathbb{R}^{n}} (K * f) (x) \varphi (x) \, dx = \int_{\mathbb{R}^{n}} (K * \hat{\varphi}) (-x) \varphi (x) \, dx.
\]
\[\square\]

**Remark 13.** The assumption \( K \in L^{2} (\mathbb{R}^{n}) \) can be dropped.

**Remark 14.** We could use standard Calderón Zygmund decomposition instead of Theorem 31.

### 5.3. Singular integrals: Some extension and variants.

**Theorem 37.** Let \( K \in L^{2} (\mathbb{R}^{n}) \). We suppose:

(a) \[ \hat{K} (x) \leq B. \]

(b) For any \( |y| > 0 \),
\[
\int_{|x| \geq 2|y|} |K (x-y) - K (x)| \, dx \leq B.
\]
For \( f \in L^{1} \cap L^{p}, 1 < p < \infty \), define
\[
(Tf) (x) = \int_{\mathbb{R}^{n}} K (x-y) f (y) \, dy.
\]
Then
\[
\|Tf\|_{p} \leq A_{p} \|f\|_{p}
\]
where \( A_{p} = c(n, p) B \).
Proof. We only need to establish weak \((1, 1)\) estimate. Let \(F, \Omega\) and \(Q_k\) be defined in the Calderon Zygmund decomposition to \(|f|\). We define \(Q^*_k = 2^{\sqrt{n}} Q_k\), \(\Omega^* = \bigcup_k Q^*_k\) and \(F^* = \mathbb{R}^n \setminus \Omega^*\).

Let \(y_k\) be the center of \(Q_k\). We have for any \(x \notin Q^*_k\) and \(y \in Q_k\),
\[
|x - y_k| \geq 2 |y - y_k|.
\]

Since \(T_k f(x) = \int_{Q_k} K(x - y)b(y)dy = \int_{Q_k} [K(x - y) - K(x - y_k)]b(y)dy\), we have for any \(x \in F^*\)
\[
\int_{F^*} |T_b(x)| \leq \sum_k \int_{x \notin Q^*_k} \int_{y \in Q_k} |[K(x - y) - K(x - y_k)]b(y)| dydx
\]
\[
= \sum_k \int_{Q_k} |b(y)| dy
\]
\[
\leq \int_{\Omega} |b(y)| dy \leq c \|f\|_1.
\]

Hence,
\[
\left| \left\{ x \in \mathbb{R}^n : |T_b(x)| > \frac{\alpha}{2} \right\} \right|
\]
\[
\leq \left| \left\{ x \in F^* : |T_b(x)| > \frac{\alpha}{2} \right\} \right| + |\Omega^*|
\]
\[
\leq \frac{c}{\alpha} \|f\|_1 + c |\Omega| \leq \frac{c}{\alpha} \|f\|_1.
\]

\[\square\]

Remark 15. Theorem 36 is a special case of the above theorem.

Theorem 38. Suppose the kernel \(K\) satisfies
(a) For any \(|x| > 0\),
\[
|K(x)| \leq B |x|^{-n}.
\]
(b) For any \(|y| > 0\),
\[
\int_{|x| \geq 2|y|} |K(x - y) - K(x)| dx \leq B.
\]
(c) For any \(0 < R_1 < R_2 < \infty\),
\[
\int_{R_1 < |x| < R_2} K(x) dx = 0
\]
For \(f \in L^p\), \(1 < p < \infty\), define for \(\varepsilon > 0\),
\[
(T_\varepsilon f)(x) = \int_{|y| \geq \varepsilon} f(x - y) K(y) dy.
\]
Then
\[
\|T_\varepsilon f\|_p \leq A_p \|f\|_p
\]
where \(A_p = c(n, p) B\). Also,
\[
\lim_{\varepsilon \to 0} T_\varepsilon f = Tf
\]
exists in $L^p$ for each $f \in L^p$ and
\[ \|Tf\|_p \leq A_p \|f\|_p. \]

**Lemma 7.** Let
\[ K_\varepsilon(x) = \begin{cases} K(x) & \text{for } |x| \geq \varepsilon, \\ 0 & \text{for } |x| < \varepsilon. \end{cases} \]
Then
\[ \sup_y \left| \hat{K}_\varepsilon(y) \right| \leq cB \]
where $c = c(n)$.

**Proof.** We first assume $\varepsilon = 1$. We observe that $K_1$ satisfies
(a) For any $|x| > 0$,
\[ |K_1(x)| \leq B |x|^{-n}. \]
(b) For any $|y| > 0$,
\[ \int_{|x| \geq 2|y|} |K_1(x - y) - K_1(x)| \, dx \leq cB. \]
(c) For any $0 < R_1 < R_2 < \infty$,
\[ \int_{R_1 < |x| < R_2} K_1(x) \, dx = 0. \]

To see this,
\[ \int_{|x| \geq 2|y|} |K_1(x - y) - K_1(x)| \, dx \]
\[ \leq \int_{|x| \geq 2|y|} |K_1(x - y) - K_1(x)| \, dx + \int_{|x| \geq 2|y|} |K_1(x - y)| \, dx + \int_{|x| \geq 2|y|} |K_1(x)| \, dx \]
\[ \leq B + \int_{1 \leq |x - y| \leq \frac{3}{2}} |K_1(x - y)| \, dx + \int_{1 \leq |x| \leq 2} |K_1(x)| \, dx \leq cB. \]

Now since $K_1 \in L^2(\mathbb{R}^n)$, for any $|y| > 0$,
\[ \hat{K}_1(y) = \lim_{R \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{|x| \leq R} K_1(x) e^{-ix \cdot y} \, dx \]
\[ = \frac{1}{(2\pi)^{n/2}} \int_{|x| \leq \frac{2\pi}{|y|}} K_1(x) e^{-ix \cdot y} \, dx + \lim_{R \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{|x| \leq \frac{2\pi}{|y|}} K_1(x) e^{-ix \cdot y} \, dx. \]

We estimate
\[ \left| \int_{|x| \leq \frac{2\pi}{|y|}} K_1(x) e^{-ix \cdot y} \, dx \right| \]
\[ = \left| \int_{|x| \leq \frac{2\pi}{|y|}} K_1(x) \left[ e^{-ix \cdot y} - 1 \right] \, dx \right| \]
\[ \leq |y| \int_{|x| \leq \frac{2\pi}{|y|}} |x| |K_1(x)| \, dx \]
\[ \leq B |y| \int_{|x| \leq \frac{2\pi}{|y|}} \frac{1}{|x|^{n-1}} \, dx \]
\[ = 2n \omega_n \pi B. \]
On the other hand, let $z = \frac{\pi y}{|y|^2}$. We have 
\[ e^{-iz \cdot y} = e^{-i \pi} = -1 \]
and 
\[ \int_{\frac{2\pi}{|y|} \leq |x| \leq R} K_1(x - z) e^{-ix \cdot y} \, dx 
= - \int_{\frac{2\pi}{|y|} \leq |x + z| \leq R} K_1(x) e^{-ix \cdot y} \, dx. \]
Hence, 
\[ \lim_{R \to \infty} \int_{\frac{2\pi}{|y|} \leq |x| \leq R} K_1(x) e^{-ix \cdot y} \, dx 
= \frac{1}{2} \lim_{R \to \infty} \int_{\frac{2\pi}{|y|} \leq |x| \leq R} \left( K_1(x) - K_1(x - z) \right) e^{-ix \cdot y} \, dx 
= \frac{1}{2} \int_{\frac{2\pi}{|y|} \leq |x| \leq 1} K_1(x) e^{-ix \cdot y} \, dx + \frac{1}{2} \int_{\frac{2\pi}{|y|} \leq |x + z|} K_1(x) e^{-ix \cdot y} \, dx 
= I + II + III. \]
We have 
\[ |II| \leq \frac{1}{2} \int_{\frac{2\pi}{|y|} \leq |x| \leq \frac{2\pi}{|y|}} |K_1(x)| \, dx \leq cB, \]
and 
\[ |III| \leq \frac{1}{2} \int_{\frac{2\pi}{|y|} \leq |x| \leq \frac{2\pi}{|y|}} |K_1(x)| \, dx \leq cB \]
and 
\[ |I| \leq \int_{\frac{2\pi}{|y|} \leq |x| \leq \infty} |K_1(x) - K_1(x - z)| \, dx = \int_{|x| \geq 2|z|} |K_1(x) - K_1(x - z)| \, dx \leq cB \]
Hence, we conclude 
\[ \sup_y \left| \hat{K}_1(y) \right| \leq cB. \]
The general $\varepsilon$ case can be obtained by a scaling.

Now we are ready to prove Theorem 38:

Proof of Theorem 38. Since 
\[ \sup_y \left| \hat{K}_\varepsilon(y) \right| \leq cB, \]
we have 
\[ \|T_\varepsilon f\|_p \leq A_p \|f\|_p \]
where $A_p = c (n, p) B$. Now assume $f \in C_0^\infty(\mathbb{R}^n)$, then we have 
\[ (T_\varepsilon f)(x) = \int_{|y| \geq \varepsilon} f(x - y) K(y) \, dy 
= \int_{|y| \geq 1} f(x - y) K(y) \, dy + \int_{\varepsilon \leq |y| \leq 1} (f(x - y) - f(x)) K(y) \, dy 
= I + II. \]
Since \( f \in L^1 \) and \( K \in L^p (\mathbb{R}^n \setminus B_1 (0)) \), \( I \in L^p \). Now \( II \) is compactly supported and since
\[
|f (x - y) - f (x)| \leq A |y|,
\]
\( II \) converges uniformly to
\[
\int_{|y| \leq 1} (f (x - y) - f (x)) \, K (y) \, dy.
\]
Hence,
\[
T_{\varepsilon} f \to \int_{|y| \geq 1} f (x - y) \, K (y) \, dy + \int_{|y| \leq 1} (f (x - y) - f (x)) \, K (y) \, dy
\]
in \( L^p \). Now standard density argument implies \( T_{\varepsilon} f \to T f \) for each \( f \in L^p \). \( \square \)

Let
\[
K (x) = \frac{1}{\pi x}, x \in \mathbb{R}^1.
\]
It is easy to check that \( K \) satisfies the assumptions in Theorem 38. Hence for any \( f \in L^p \), the Hilbert transform
\[
T f = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f (x - y)}{y} \, dy
\]
eexists in the \( L^p \) norm.

5.3.1. **Singular integral operators which commute with dilations.** Let
\[
T f = K \ast f
\]
for \( f \in L^p \). \( T \) commutes with dilation iff \( K \) is homogeneous of degree \(-n\). i.e.,
\[
K (x) = \frac{\Omega (x)}{|x|^n},
\]
where \( \Omega \) is homogeneous of degree \(0\), i.e.,
\[
\Omega (\varepsilon x) = \Omega (x) \text{ for any } \varepsilon > 0.
\]
When \( K \) is homogeneous of degree \(-n\), the cancellation property in Theorem 38 is equivalent to
\[
\int_{\mathbb{S}^{n-1}} \Omega (x) \, d\sigma = 0.
\]
For the assumption \( b \), we have

**Lemma 8.** Assume that \( \Omega \) satisfies the following "Dini-type" condition
\[
(5.1) \quad \int_0^1 \frac{\omega (\delta)}{\delta} \, d\delta < \infty
\]
where the modulus of continuity
\[
\omega (\delta) = \sup_{|x - x'| \leq \delta, \ |x| = |x'| = 1} |\Omega (x) - \Omega (x')|.
\]
Then there exists \( B > 0 \), such that for any \( y > 0 \),
\[
\int_{|x| \geq 2|y|} |K (x - y) - K (x)| \, dx \leq B.
\]
Proof.

\[ K(x - y) - K(x) = \frac{\Omega(x - y) - \Omega(x)}{|x - y|^n} + \Omega(x) \left[ \frac{1}{|x - y|^n} - \frac{1}{|x|^n} \right]. \]

Now

\[ \int_{|x| \geq 2|y|} \left| \frac{1}{|x - y|^n} - \frac{1}{|x|^n} \right| \, dx \leq c. \]

Since

\[ \left| \frac{x - y}{|x - y|} - \frac{x}{|x|} \right| \leq c \frac{|y|}{|x|} \]

holds for any \(|x| \geq 2|y|\), we have

\[ |\Omega(x - y) - \Omega(x)| = \left| \Omega \left( \frac{x - y}{|x - y|} \right) - \Omega \left( \frac{x}{|x|} \right) \right| \leq \omega \left( c \frac{|y|}{|x|} \right) \]

and

\[
\int_{|x| \geq 2|y|} \left| \frac{\Omega(x - y) - \Omega(x)}{|x - y|^n} \right| \, dx \\
\leq c \int_{|x| \geq 2|y|} \omega \left( c \frac{|y|}{|x|} \right) \frac{1}{|x|^n} \, dx \\
= c \int_{0}^{\frac{2}{c}} \omega(\delta) \frac{d\delta}{\delta} < \infty.
\]

Hence,

\[
\int_{|x| \geq 2|y|} |K(x - y) - K(x)| \, dx \leq c \int_{0}^{\frac{2}{c}} \omega(\delta) \frac{d\delta}{\delta} + c \|\Omega\|_{\infty} \\
\leq c \left( \int_{0}^{1} \frac{\omega(\delta)}{\delta} \, d\delta + \|\Omega\|_{\infty} \right). \]

\[ \square \]

Remark 16. The Dini-type condition is satisfied if \( \Omega \) is Hölder continuous.

Theorem 39. Let \( \Omega \) be homogeneous of degree 0. Suppose that

\[ \int_{\mathbb{R}^{n-1}} \Omega(x) \, d\sigma = 0 \]

and Dini condition (5.1) holds. For \( 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^n) \), define

\[ T_\varepsilon f = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x - y) \, dy. \]

(a) There exists constant \( A_p \) independent of \( \varepsilon \), s.t.

\[ \|T_\varepsilon f\|_p \leq A_p \|f\|_p. \]

(b) \[ \lim_{\varepsilon \to 0} T_\varepsilon f = Tf \]

exists in \( L^p \) norm, and

\[ \|Tf\|_p \leq A_p \|f\|_p. \]
(c) If \( f \in L^2 (\mathbb{R}^n) \), then the Fourier transforms of \( f \) and \( T f \) are related by

\[
\hat{T f} (y) = m (y) \hat{f} (y)
\]

where \( m \) is a homogeneous function of degree 0.

Proof. (a) (b) follows from Theorem 38.

(c) Let

\[
K_{\varepsilon, \eta} (x) = \begin{cases} \frac{\Omega (x)}{|x|} & \text{if } \varepsilon \leq |x| \leq \eta, \\ 0 & \text{otherwise.} \end{cases}
\]

Then we have

\[
\hat{K}_{\varepsilon, \eta} (y) = \frac{1}{(2\pi)^{n/2}} \int_{\varepsilon \leq |x| \leq \eta} \frac{\Omega (x)}{|x|^{n/2}} e^{-i x \cdot y} dx.
\]

Writing \( r = |x|, R = |y|, x = |x| x' \) and \( y = |y| y' \) for \( x, y \neq 0 \), we have for any \( y \neq 0 \),

\[
\hat{K}_{\varepsilon, \eta} (y) = \frac{1}{(2\pi)^{n/2}} \int_{\varepsilon \leq |x'| \leq \eta} \frac{\Omega (x')}{r^n} e^{-i R x' \cdot y'}dx'
\]

where

\[
I_{\varepsilon, \eta} (x', y) = \int_{\varepsilon}^{\eta} \frac{1}{r} \left( e^{-i R x' \cdot y'} - \cos (R r) \right) dr.
\]

Now,

\[
\text{Im} I_{\varepsilon, \eta} = - \int_{\varepsilon}^{\eta} \frac{1}{r} \sin (R x' \cdot y') dr
\]

\[
\rightarrow - \int_{0}^{\infty} \frac{\sin t}{t} dt \text{ sign (} x' \cdot y' \text{)} = - \frac{\pi}{2} \text{ sign (} x' \cdot y' \text{)}
\]

as \( \varepsilon \to 0 \) and \( \eta \to \infty \). Moreover, \( |\text{Im} I_{\varepsilon, \eta}| \) is uniformly bounded for \( 0 < \varepsilon < \eta < \infty \).

On the other hand,

\[
\text{Re} I_{\varepsilon, \eta} = \int_{\varepsilon}^{\eta} \frac{1}{r} \left[ \cos (R x' \cdot y') - \cos (R r) \right] dr
\]

\[
\rightarrow \ln \frac{1}{|x' \cdot y'|}
\]
as \( \varepsilon \to 0, \eta \to \infty \) and \( \text{Re} I_{\varepsilon, \eta} \leq 2 \ln \frac{1}{|x' \cdot y'|} \). To see this, we consider for \( \lambda, \mu > 0 \),

\[
\int_{\varepsilon}^{\eta} \frac{\cos (\lambda r) - \cos (\mu r)}{r} \, dr
= \int_{\lambda \eta}^{\lambda \varepsilon} \frac{\cos r}{r} \, dr - \int_{\mu \varepsilon}^{\mu \eta} \frac{\cos r}{r} \, dr
= \int_{\mu \eta}^{\lambda \eta} \frac{\cos r}{r} \, dr + \int_{\lambda \varepsilon}^{\mu \varepsilon} \frac{\cos r}{r} \, dr
= \int_{\mu}^{\lambda} \frac{1}{r} \cos \left( r \frac{r}{\eta} \right) \, dr + \int_{\lambda \varepsilon}^{\mu \varepsilon} \frac{\cos r}{r} \, dr + \int_{\lambda \varepsilon}^{\mu \varepsilon} \frac{\cos r - 1}{r} \, dr
= \int_{\mu}^{\lambda} \frac{1}{r} \cos \left( r \frac{r}{\eta} \right) \, dr + \ln \frac{\mu}{\lambda} + \int_{\lambda \varepsilon}^{\mu \varepsilon} \frac{\cos r - 1}{r} \, dr.
\]

Since

\[
\left| \int_{\lambda \varepsilon}^{\mu \varepsilon} \frac{\cos r - 1}{r} \, dr \right| \leq |\mu - \lambda| \varepsilon,
\]

we have

\[
\lim_{\eta \to \infty} \int_{\varepsilon}^{\eta} \frac{\cos (\lambda r) - \cos (\mu r)}{r} \, dr = \ln \frac{\mu}{\lambda}.
\]

Moreover

\[
\left| \int_{\varepsilon}^{\eta} \frac{\cos (\lambda r) - \cos (\mu r)}{r} \, dr \right|
\leq \left| \int_{\mu \eta}^{\lambda \eta} \frac{\cos r}{r} \, dr \right| + \left| \int_{\lambda \varepsilon}^{\mu \varepsilon} \frac{\cos r}{r} \, dr \right| \leq 2 \ln \frac{\mu}{\lambda}.
\]

Hence, we have

\[
\lim_{\varepsilon \to 0, \eta \to \infty} I_{\varepsilon, \eta}(x', y) = -\frac{\pi i}{2} \text{sign} (x' \cdot y') + \ln \frac{1}{|x' \cdot y'|}
\]

and for any \( 0 < \varepsilon < \eta < \infty \),

\[
|I_{\varepsilon, \eta}(x', y)| \leq c \left( \ln \frac{1}{|x' \cdot y'|} + 1 \right)
\]

for some universal constant \( c \). Since for any \( |y'| = 1 \),

\[
\int_{\mathbb{S}^{n-1}} \frac{1}{|x' \cdot y'|} \, d\sigma(x') \leq c,
\]

we have for some \( c = c(n) \),

\[
\sup_{0 < \varepsilon < \eta < \infty} \left\| \hat{K}_{\varepsilon, \eta}(y) \right\|_{\infty} \leq c,
\]

and Lebesgue dominated convergence implies

\[
\lim_{\varepsilon \to 0, \eta \to \infty} \hat{K}_{\varepsilon, \eta}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^{n-1}} \Omega(x') I(x', y') \, d\sigma(x')
\]

where

\[
I(x', y') = -\frac{\pi i}{2} \text{sign} (x' \cdot y') + \ln \frac{1}{|x' \cdot y'|}.
\]
Now for any \( f \in L^2 \), we have
\[
\hat{T_{\varepsilon,\eta}f}(y) = (2\pi)^{-\frac{n}{2}} \hat{K}_{\varepsilon,\eta}(y) \to m(y) \hat{f}(y)
\]
in \( L^2 \) as \( \varepsilon \to 0, \eta \to \infty \), where
\[
m(y) = \int_{S^{n-1}} \Omega(x') I(x', y') \, d\sigma(x').
\]
Hence, \( T_{\varepsilon,\eta}f \) converges to some function \( g \in L^2 \). Now fix \( \varepsilon \), it is easy to see \( T_{\varepsilon,\eta}f \to Tf \) in \( L^2 \). So we conclude \( g = Tf \). Hence
\[
\hat{Tf}(y) = m(y) \hat{f}(y).
\]
\[\square\]

From the proof, we see
\[
m(y) = \int_{S^{n-1}} \Omega(x') I(x', y') \, d\sigma(x')
\]
with
\[
I(x', y') = -\frac{\pi i}{2} \text{sign} (x' \cdot y') + \ln \frac{1}{|x' \cdot y'|}.
\]
In particular, for the kernel \( \frac{1}{\pi x} \) of Hilbert transform, we have
\[
m(y) = \int_{S^{n-1}} -\frac{i}{2} x' \text{sign} (x' \cdot y') \, d\sigma(x') = -i \text{sign} (y).
\]

**Theorem 40.** Suppose \( \Omega \) satisfies the conditions of the previous theorem. Let \( f \in L^p(\mathbb{R}^n), 1 \leq p < \infty \).
(a) \( \lim_{\varepsilon \to 0} T_{\varepsilon}f(x) \) exists for a.e. \( x \);
(b) Let
\[
T^*f(x) = \sup_{\varepsilon > 0} |T_{\varepsilon}f(x)|.
\]
Then the map \( f \mapsto T^*f \) is of weak type \( (1,1) \).
(c) If \( 1 < p < \infty \), then \( \|T^*f\|_p \leq A_p \|f\|_p \).

**Proof.** Check the proof of (b) and (c) in Stein’s book if you are interested. We now prove (a) assuming that (b) and (c) hold.

We define for any \( f \in L^p(\mathbb{R}^n) \),
\[
A f(x) = \left| \limsup_{\varepsilon \to 0} T_{\varepsilon}f(x) - \liminf_{\varepsilon \to 0} T_{\varepsilon}f(x) \right|.
\]
Then \( A f(x) \leq 2T^*f(x) \). We also observe that if \( f \in \mathcal{D}(\mathbb{R}^n) \), then \( A f(x) \equiv 0 \). If \( 1 < p < \infty \), writing \( f = f_1 + f_2, \) where \( f_1 \in \mathcal{D}(\mathbb{R}^n) \), we have
\[
\|A f\|_p = \|A f_2\|_p \leq 2\|T^*f_2\|_p \leq 2A_p \|f_2\|_p.
\]
Since we can make \( \|f_2\|_p \) arbitrarily small, we have \( \|A f\|_p = 0 \). Hence, \( A f(x) = 0 \) a.e. in \( \mathbb{R}^n \). If \( p = 1 \), we have for any \( t > 0 \),
\[
\{|A f > t|\} = |\{A f_2 > t\}| \leq \left| \left\{ T^*f_2 > \frac{t}{2} \right\} \right|
\leq \frac{2t}{t} \|f_2\|_1.
\]
Since we can make \( \|f_2\|_1 \) arbitrarily small, we have \( \{|A f > t|\} = 0 \) for every \( t > 0 \), i.e., \( A f(x) = 0 \) a.e. in \( \mathbb{R}^n \). \[\square\]
5.4. Hilbert transform on a circle and Fourier series. For any \( f \in L^p [-\pi, \pi] \), we can define the Hilbert transform

\[
Sf(y) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \frac{\cot (x/2)}{2\pi} f(x - y) \, dx.
\]

Comparing this with the modified Hilbert transform

\[
\tilde{S}f(y) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \frac{1}{\pi x} f(x - y) \, dx.
\]

We see that \( Sf(y) \) exists for a.e. \( y \in [-\pi, \pi] \) and \( S \) is of type \((p, p)\) for any \( 1 < p < \infty \).

Lemma 9. For any \( n \in \mathbb{Z} \),

\[
S(e^{inx}) = -i \text{sign} (n) e^{inx}.
\]

Proof. Since \( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin ((N+\frac{1}{2})y)}{\sin \frac{y}{2}} dy = 1 \) for \( N = 0, 1, 2, \ldots \), we have

\[
(S(e^{inx})) y = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \frac{\cos (x/2)}{2\pi \sin (x/2)} e^{in(y-x)} \, dx
\]

\[
= -i e^{iny} \int_{-\pi}^{\pi} \frac{\cos (x/2)}{2\pi \sin (x/2)} \sin (nx) \, dx
\]

\[
= -i e^{iny} \int_{-\pi}^{\pi} \frac{\sin ((n + \frac{1}{2})x) + \sin ((n - \frac{1}{2})x)}{4\pi \sin (x/2)} \, dx
\]

\[
= -i \text{sign} (n) e^{iny}.
\]

Hence, if

\[
f(x) = \sum a_n e^{inx},
\]

then

\[
iSf(y) = \sum_{n \geq 1} a_n e^{inx} - \sum_{n \leq -1} a_n e^{inx}.
\]

We define

\[
P_0 f = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = a_0(f),
\]

then

\[
Tf = \frac{1}{2} (iS + I + P_0) f = \sum_{n \geq 0} a_n e^{inx}.
\]

We also define for any integer \( k \),

\[
M_k f = e^{ikx} f
\]

It is easy to verify that

\[
s_N = M_{-N}TM_N - M_{N+1}TM_{-N+1}.
\]

Now we apply the singular integral theory to prove the \( L^p \) convergence of Fourier series:

Theorem 41. Let \( f \in L^p [-\pi, \pi] \), \( 1 < p < \infty \), then \( s_N(f) \to f \) in \( L^p [-\pi, \pi] \) as \( N \to \infty \).
Proof. Since $S$ is of type $(p,p)$ and $P_0$ is of type $(p,p)$, we conclude $T$ is of type $(p,p)$. Obviously $M_N$ is of type $(p,p)$, hence $s_N$ is of type $(p,p)$. Since $s_N(f) \to f$ in $L^p[-\pi, \pi]$ as $N \to \infty$ for $f \in C^2(S^1)$, standard density argument implies that $s_N(f) \to f$ in $L^p[-\pi, \pi]$ as $N \to \infty$ for every $f \in L^p[-\pi, \pi]$. \hfill \Box

6. RIESZ TRANSFORMS, POISSON INTEGRALS AND SPHERICAL HARMONICS

6.1. The Riesz transforms. Recall that for $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, the Hilbert transform of $f$ is defined by

$$Hf(y) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} dy, \ y \in \mathbb{R}.$$ 

Since

$$\widehat{Hf} = -i \text{sign} (y) \hat{f},$$

Hilbert transform is unitary on $L^2(\mathbb{R})$ and $H^2 = -I$.

For any $\delta \neq 0$, we define the dilation operator $\tau_\delta (f)(x) = f(\delta x)$. For any $\delta > 0$, $H\tau_\delta = \tau_\delta H$ and for any $\delta < 0$, $H\tau_\delta = -\tau_\delta H$.

**Theorem 42.** Suppose $T$ is a bounded operator on $L^2(\mathbb{R})$ which satisfies

(a) $T$ commutes with translations;

(b) for any $\delta > 0$, $T\tau_\delta = \tau_\delta T$ and for any $\delta < 0$, $T\tau_\delta = -\tau_\delta T$.

Then $T$ is a constant multiple of the Hilbert transform.

Proof. There exists $m \in L^\infty(\mathbb{R})$, s.t.

$$\widehat{Tf}(y) = m(y) \hat{f}(y) \text{ for a.e. } y \in \mathbb{R}.$$ 

Since

$$\mathcal{F}\tau_\delta = |\delta|^{-1} \tau_{|\delta|^{-1}} \mathcal{F},$$

we have

$$\widehat{T\tau_\delta f}(y) = m(y) \tau_\delta \hat{f}(y) = m(y) |\delta|^{-1} \tau_{|\delta|^{-1}} \hat{f}(y) = m(y) |\delta|^{-1} \hat{f}(\delta^{-1}y),$$

and

$$\widehat{T\tau_\delta f}(y) = \text{sign}(\delta) \tau_\delta \widehat{Tf}(y) = \text{sign}(\delta) |\delta|^{-1} \tau_{|\delta|^{-1}} \widehat{Tf}(y)$$

$$= \text{sign}(\delta) |\delta|^{-1} \widehat{Tf}(\delta^{-1}y) = \text{sign}(\delta) |\delta|^{-1} m(\delta^{-1}y) \hat{f}(\delta^{-1}y).$$

Hence

$$m(y) = \text{sign}(\delta) m(\delta^{-1}y)$$

which implies that for some constant $c$, $m(y) = -ci \text{sign}(y)$ a.e.. \hfill \Box

Now we generalize the Hilbert transform to multi-dimensions. For any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the Riesz transform of $f$ is defined by

$$R_j f(x) = \lim_{\varepsilon \to 0} c_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y) dy, \ j = 1, 2, \cdots, n$$

where

$$c_n = \frac{\Gamma (n+1)}{\pi^{(n+1)/2}}.$$
For the Riesz kernel
\[ K_j(x) = \frac{\Omega_j(x)}{|x|^{n+1}} \] and \( \Omega_j(x) = c_n \frac{x_j}{|x|^n} \), we have
\[
m(y) = \int_{S^{n-1}} \Omega_j(x') I(x', y') \, d\sigma(x')
\]
\[
= c_n \int_{S^{n-1}} x' \left[ -\frac{\pi i}{2} \text{sign}(x' \cdot y') + \ln \frac{1}{|x' \cdot y'|} \right] \, d\sigma(x')
\]
\[
= -\frac{\pi i}{2} c_n \int_{S^{n-1}} x' \text{sign}(x' \cdot y') \, d\sigma(x').
\]
Since \( m \) commutes with rotations, we have \( m(y) = c \frac{y}{|y|} \). Choose \( y = e_n \), we have for \( n \geq 2 \),
\[
c = -\frac{\pi i}{2} c_n \int_{S^{n-1}} x'_n \text{sign}(x'_n) \, d\sigma(x')
\]
\[
= -\frac{\pi i}{2} c_n \int_{S^{n-1}} |x'_n| \, d\sigma(x')
\]
\[
= -\pi i c_n \int_0^1 t (1 - t^2)^{\frac{n-2}{2}} (n-1) \omega_{n-1} \frac{dt}{\sqrt{1 - t^2}}
\]
\[
= -\pi i \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2} \omega_{n-1}}
\]
\[
= -\pi i \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2} \Gamma \left( \frac{n+1}{2} \right)}
\]
\[
= -i.
\]
Here we used
\[
\omega_n = \frac{\pi^{n/2}}{\Gamma \left( 1 + \frac{n}{2} \right)}.
\]
Hence,
\[
\hat{R}_j f(y) = -i \frac{y_j}{|y|} \hat{f}(y).
\]
To understand the interaction of dilations and rotations with the Riesz transform, we consider multipliers which commute with dilations and rotations.

**Lemma 10.** Suppose \( m : \mathbb{R}^n \to \mathbb{R}^n \) is homogeneous of degree 0, and for any rotation \( \rho \),
\[
m(\rho x) = \rho m(x).
\]
Then
\[
m(x) = c \frac{x}{|x|}
\]
for some constant \( c \).

**Proof.** Fix any unit vector \( e \), we have
\[
m(\rho e) = \rho m(e).
\]
Hence, \( \rho e = e \) implies \( \rho m(e) = m(e) \), so there exists constant \( c(e) \) s.t. \( m(e) = c(e) e \). Now let \( e_1, e_2 \) be two unit vectors, and \( \rho \) be a rotation such that \( \rho e_1 = e_2 \), then
\[
m(\rho e_1) = \rho m(e_1) = c(e_1) e_2,
\]
\[
m(\rho e_1) = m(e_1) = c(e_2) e_2,
\]
hence \( c(e_1) = c(e_2) \). Hence \( m(e) = ce \). Since \( m \) is homogeneous of degree 0,

\[ m(x) = c \frac{x}{|x|} \]

for some constant \( c \).

To translate the above lemma in terms of Riesz transforms, we consider the interaction of dilations and rotations with Fourier transform. For any \( \delta > 0 \), we have

\[ \mathcal{F}_\tau \delta = \delta^{-n} \tau_\delta^{-1} \mathcal{F}. \]

And for any rotation about the origin \( \rho \), we define \( \rho f(x) = f(\rho x) \), then

\[
(F \rho f)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot y} f(\rho x) \, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\rho^{-1}x \cdot y} f(x) \, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \rho y} f(x) \, dx = (\rho F f)(y).
\]

**Proposition 8.** Let \( T = (T_1, T_2, \cdots, T_n) \) be an \( n \)-tuple of bounded transformations on \( L^2(\mathbb{R}^n) \). Suppose

(a) Each \( T_j \) commutes with the translation in \( \mathbb{R}^n \);
(b) Each \( T_j \) commutes with the dilations in \( \mathbb{R}^n \);
(c) For every rotation \( \rho \) of \( \mathbb{R}^n \),

\[ \rho T_j \rho^{-1} f = \sum_{k=1}^{n} \rho_{jk} T_k f. \]

Then there exists constant \( c \) such that \( T_j = c R_j \), \( j = 1, 2, \cdots, n \).

We now consider a function \( u \in C^2(\mathbb{R}^n) \) with compact support. We have

\[ u_{x_i x_j} = -y_i y_j \hat{u} = -y_i y_j \frac{|y|}{|y|} |y|^2 \hat{u} = -R_i R_j \Delta u. \]

Hence,

\[ \frac{\partial^2 u}{\partial x_i \partial x_j} = -R_i R_j \Delta u. \]

i.e., once we know \( \Delta u \), we know all second order partial derivatives of \( u \).

**Theorem 43.** Suppose \( u \in C^2(\mathbb{R}^n) \) is compact supported. Then for any \( 1 < p < \infty \), we have

\[ \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_p \leq A_p \| u \|_p \]

where \( A_p \) is a constant depending only on \( n \) and \( p \).
6.2. Poisson Integral. We write $\mathbb{R}^{n+1}_+ = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$. Given a function $f$ defined on $\mathbb{R}^n$, we want to find harmonic function $u$ on $\mathbb{R}^{n+1}_+$ such that $u = f$ on $\mathbb{R}^n$. We use $\Delta$ to denote the Laplacian on $\mathbb{R}^n$, and $\tilde{\Delta}$ the Laplacian on $\mathbb{R}^{n+1}$. Then

$$\tilde{\Delta} u = \Delta u + u_{tt} = 0.$$ 

We write $\hat{u} = \hat{u}(y, t)$ the Fourier transform of $u$ in $x$-variable, then

$$-|y|^2 \hat{u} + \hat{u}_{tt} = 0.$$ 

We also impose boundary condition $\hat{u}(y, 0) = \hat{f}$ and $\hat{u}(y, \infty) = 0$. Then formally

$$\hat{u} = e^{-|y|t} \hat{f}.$$ 

Hence,

$$u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ixy} e^{-|y|t} \hat{f}(y) dy.$$ 

If $f \in L^2(\mathbb{R}^n)$, $u$ defined above is a smooth harmonic function in $\mathbb{R}^{n+1}_+$ and $u(\cdot, t) \to f$ in $L^2(\mathbb{R}^n)$ norm as $t \to 0^+$. Hence, indeed $u$ is a solution to the Dirichlet problem in $\mathbb{R}^{n+1}_+$.

Now we write

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ixy} e^{-|y|t} \left( \int_{\mathbb{R}^n} e^{-iyz} f(z) dz \right) dy$$

where

$$P(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ixy} e^{-|y|t} dy$$

is called the Poisson kernel in $\mathbb{R}^{n+1}_+$.

**Lemma 11.** For any $\gamma > 0$,

$$e^{-\gamma} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\gamma^2}{4u}} du.$$ 

**Proof.**

$$e^{-\gamma} = \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{is}}{1 + s^2} ds = \frac{1}{\pi} \int_{-\infty}^\infty e^{is} \left( \int_0^\infty e^{-(1+s^2)u} du \right) ds.$$ 

$$\square$$

**Proposition 9.**

$$P(x, t) = \frac{c_n t}{\left(|x|^2 + \epsilon^2 t^2\right)^{\frac{n+1}{2}}} \quad \text{with} \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$
Proof.

\[ P(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot y} e^{-|y|^2/4t} dy \]

\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot y} \left( \int_0^\infty e^{-u} e^{-|y|^2/4u} du \right) dy \]

\[ = \frac{1}{(2\pi)^n} \frac{1}{\sqrt{t}} \int_0^\infty e^{-u} \left( \int_{\mathbb{R}^n} e^{ix \cdot y} e^{-|y|^2/4u} dy \right) du. \]

Now

\[ \int_{\mathbb{R}^n} e^{ix \cdot y} e^{-|y|^2/4u} dy \]

\[ = \int_{\mathbb{R}^n} e^{-\sqrt{u} \cdot iy} e^{-\sqrt{u} \cdot x} e^{-|y|^2/4} dy \]

\[ = e^{-\frac{|x|^2 u}{4}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4u}} dy \]

\[ = e^{-\frac{|x|^2 u}{4}} \left( \frac{2\sqrt{u}}{t} \right)^n \left( \int_{-\infty}^{\infty} e^{-s^2} ds \right)^n \]

\[ = e^{-\frac{|x|^2 u}{4}} \left( \frac{2\sqrt{\pi u}}{t} \right)^n. \]

Hence

\[ P(x, t) = \frac{1}{(2\pi)^n} \frac{(2\sqrt{\pi})^n}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{-n/2} e^{-\frac{|x|^2 u}{4u}} du \]

\[ = \frac{1}{\pi^{n/2}} \left( \frac{t}{|x|^2 + t^2} \right)^{n/2} \int_0^\infty e^{-s} s^{n-1} ds \]

\[ = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{n/2}} \left( \frac{t}{|x|^2 + t^2} \right)^{n/2}. \]

\[ \square \]

**Proposition 10.** The Poisson kernel \( P(x, t) \) satisfies:

(i) \( P > 0 \) for any \( x \in \mathbb{R}^n, t > 0; \)

(ii) for any \( t > 0, \)

\[ \int_{\mathbb{R}^n} P(x, t) dx = 1. \]

(iii) For any \( \delta > 0, \)

\[ \lim_{t \to 0^+} \int_{|x| \geq \delta} P(x, t) dx = 0. \]

**Proof.** Direct verification. \( \square \)

**Theorem 44.** Suppose \( f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty, \) and let \( u(x, t) = P(\cdot, t) * f. \)

(i) \( u \) is a smooth harmonic function in \( \mathbb{R}^{n+1}_+; \)

(ii) for each \( x \in \mathbb{R}^n, \)

\[ \sup_{t > 0} |u(x, t)| \leq M f(x); \]
(iii) If $p < \infty$, $u(\cdot, t) \to f$ in $L^p(\mathbb{R}^n)$ as $t \to 0^+$;
(iv) For a.e. $x \in \mathbb{R}^n$,
$$\lim_{t \to 0^+} u(x, t) = f(x);$$
(v) If $p = \infty$ and $f$ is continuous in $\mathbb{R}^n$, then $u(x, t) \to f(x)$ uniformly on any compact subset of $\mathbb{R}^n$ as $t \to 0^+$.

Proof. (i) Trivial.
(ii) Assume $x = 0$ and $Mf(0) < \infty$. Define
$$g(r) = \int_{B_r(0)} |f(x)| \, dx \leq \omega_n r^n Mf(0),$$
then
$$|u(0, t)| = \left| \int_{\mathbb{R}^n} P(-x, t) f(x) \, dx \right| \leq \int_{\mathbb{R}^n} P(-x, t) |f(x)| \, dx$$
$$= \int_0^\infty P(r, t) \left( \int_{\partial B_r} |f(x)| \, d\sigma(x) \right) \, dr$$
$$= \int_0^\infty P(r, t) g(r) \, dr = g(r) P(r, t) \left|_0^\infty \right. - \int_0^\infty P_r(r, t) g(r) \, dr$$
$$= - \int_0^\infty P_r(r, t) g(r) \, dr \leq -Mf(0) \int_0^\infty P_r(r, t) \omega_n r^n \, dr$$
$$= -Mf(0) \omega_n r^n P(r, t) \left|_0^\infty \right. + Mf(0) \int_0^\infty P(r, t) n \omega_n r^{n-1} \, dr$$
$$= Mf(0).$$
Here we used the fact that $P$ is radial and monotone decreasing in $|x|$.
(iii) It follows from the fact that $P(\cdot, t)$ is an approximate delta function as $t \to 0^+$.
(iv) First assume $1 \leq p < \infty$. Define for any $x$,
$$\Omega f(x) = \limsup_{t \to 0^+} u(x, t) - \liminf_{t \to 0^+} u(x, t).$$
Then
$$|\Omega f(x)| \leq 2Mf(x)$$
and hence for any $\delta > 0$,
$$|\{ x : \Omega f(x) > \delta \}| \leq \left| \left\{ x : \Omega f(x) > \frac{\delta}{2} \right\} \right| \leq A_p \| f \|_p.$$
i.e., \( \lim_{t \to 0^+} u(x,t) \) exists a.e. \( x \in \mathbb{R}^n \). Because of (iii)
\[
\lim_{t \to 0^+} u(x,t) = f(x)
\]
a.e.

Now for \( f \in L^\infty(\mathbb{R}^n) \), if suffices to show that for any \( R > 0, \)
\[
\lim_{t \to 0^+} u(x,t) = f(x)
\]
a.e. \( x \in B_R(0) \subset \mathbb{R}^n \). We write \( f = f_1 + f_2 \) where \( f_1 = f \chi_{B_{2R}} \), then
\[
\lim_{t \to 0} f_2 * P = 0
\]
uniformly on \( B_R(0) \). and the result follows from the \( L^1 \) case for \( f_1 \).

(v) Similar. \( \square \)

6.3. Spherical harmonics. Let \( \mathcal{P}_k, k \geq 0 \) be the set of all homogeneous polynomials in \( \mathbb{R}^n \) of order \( k \). Hence \( p \in \mathcal{P}_k \) iff
\[
p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha.
\]
\( \mathcal{P}_k \) is a vector space of dimension
\[
d_k = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!k!}.
\]
We introduce an inner product on \( \mathcal{P}_k \) by
\[
\langle p, q \rangle = p(D) \bar{q}.
\]

**Definition 12.** A solid spherical harmonic of degree \( k \) is a homogeneous harmonic polynomial in \( \mathbb{R}^n \) of order \( k \). A (surface) spherical harmonic of degree \( k \) is the restriction on \( S^{n-1} \) of solid spherical harmonic of degree \( k \) in \( \mathbb{R}^n \). We use \( \mathcal{H}_k \) to denote all spherical harmonics of degree \( k \) and we will not distinguish solid spherical harmonic and surface spherical harmonic.

**Lemma 12.** For \( k \geq 2 \), the linear map
\[
\varphi : \mathcal{P}_k \to \mathcal{P}_{k-2}
\]
defined by \( \varphi(p) = \Delta p \) is onto.

**Proof.** If not, there exists \( 0 \neq q \in \mathcal{P}_{k-2}, q \perp \Delta p \) for any \( p \in \mathcal{P}_k \), hence
\[
0 = \langle q, \Delta p \rangle = \langle |x|^2 q, p \rangle
\]
which is a contradiction if we take \( p = |x|^2 q \). \( \square \)

**Remark 17.** As a direct consequence of the lemmas, for \( k \geq 2 \), the dimension of \( \mathcal{H}_k \) is
\[
d_k - d_{k-2} = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}.
\]
Moreover, the dimension of \( \mathcal{H}_0 \) is 1 and the dimension of \( \mathcal{H}_1 \) is \( n \). When \( n = 3 \), the dimension of \( \mathcal{H}_k \) is \( 2k+1 \) for every \( k \geq 0 \).
Remark 18. Let $\Delta_S$ denote the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$, i.e., the spherical Laplacian. Then we have for any $f$ defined on $\mathbb{S}^{n-1}$,

$$\Delta_S f = \Delta g$$

where $g = f \left( \frac{x}{|x|} \right)$. In polar coordinates, we have for any $f$ defined on $\mathbb{R}^n$,

$$\Delta f = f_{rr} + \frac{n-1}{r} f_r + \frac{1}{r^2} \Delta_S f.$$ 

Hence, if $p \in \mathcal{H}_k$, we have

$$\Delta_S p = -k (k + n - 2) p.$$

i.e., $p$ is an eigenfunction of the spherical Laplacian.

**Theorem 45.** If $p \in \mathcal{P}_k$, then

$$p(x) = p_0(x) + |x|^2 p_1(x) + \cdots + |x|^{2l} p_l(x)$$

where $l = \left[ \frac{k}{2} \right]$ and $p_j(x) \in \mathcal{H}_{k-2j}$.

**Proof.** Since any polynomial of order $\leq 1$ is harmonic, we can assume $k \geq 2$. We claim $\mathcal{P}_k$ is the direct sum of $|x|^2 \mathcal{P}_{k-2}$ and $\mathcal{H}_k$. For any $q \in \mathcal{P}_k$ s.t.

$$0 = \langle |x|^2 p, q \rangle$$

holds for any $p \in \mathcal{P}_{k-2}$, we have

$$0 = \langle p, \Delta q \rangle,$$

hence $\Delta q = 0$ and $q \in \mathcal{H}_k$.

The theorem then follows from induction. \hfill \Box

**Corollary 7.** The restriction of any polynomial on $\mathbb{S}^{n-1}$ is a sum of spherical harmonics.

**Corollary 8.** The collection of finite linear combinations of spherical harmonics is dense in $C \left( \mathbb{S}^{n-1} \right)$ and $L^2 \left( \mathbb{S}^{n-1} \right)$.

**Proof.** Density in $C \left( \mathbb{S}^{n-1} \right)$ follows from Weierstrass theorem. Density in $L^2 \left( \mathbb{S}^{n-1} \right)$ follows from density in $C \left( \mathbb{S}^{n-1} \right)$. \hfill \Box

The inner product in $L^2 \left( \mathbb{S}^{n-1} \right)$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{S}^{n-1}} f(x) \overline{g}(x) d\sigma(x).$$

**Theorem 46.** The finite dimensional spaces $\{\mathcal{H}_k\}_{k=0}^\infty$ are mutually orthogonal in $L^2 \left( \mathbb{S}^{n-1} \right)$. Hence

$$L^2 \left( \mathbb{S}^{n-1} \right) = \sum_{k=0}^\infty \mathcal{H}_k.$$
Proof. If \( k \neq j \) and \( p \in \mathcal{H}_k, \ q \in \mathcal{H}_j \), then
\[
(k - j) (p, q) = (k - j) \int_{S^{n-1}} p (x) \bar{q} (x) \, d\sigma (x)
= \int_{S^{n-1}} \frac{\partial p}{\partial \nu} \bar{q} - \frac{\partial \bar{q}}{\partial \nu} p \, d\sigma (x)
= \int_{|x| \leq 1} \Delta p \bar{q} - \Delta \bar{q} p \, d\sigma (x) = 0.
\]
Hence \((p, q) = 0. \)

When \( n = 2 \), the above decomposition gives rise to the Fourier series on \( S^1 \).

**Proposition 11.** Suppose \( f \in L^2 (S^{n-1}) \) and
\[
f = \sum_{k=0}^{\infty} p_k
\]
where \( p_k \in \mathcal{H}_k \). Then \( f \) is smooth iff for any \( N > 0, \)
\[
\int_{S^{n-1}} |p_k|^2 = O (k^{-N})
\]
as \( k \to \infty \).

Proof. If \( f \) is smooth, for any positive integer \( r, \)
\[
\int p_k \Delta_r^* f = \int \Delta_r^* p_k f = \left[ -k (k + n - 2) \right]^r \int |p_k|^2,
\]
hence for \( k \) large,
\[
\int |p_k|^2 \leq \frac{1}{|k (k + n - 2)|^r} \left( \int |p_k|^2 \right)^{\frac{1}{2}} \left( \int |\Delta_r^* f|^2 \right)^{\frac{1}{2}} \leq \frac{c \left( \int |p_k|^2 \right)^{\frac{1}{2}}}{k^{2r}}.
\]
So we have
\[
\int |p_k|^2 \leq \frac{c}{k^{4r}}.
\]
On the other hand, if for any \( N > 0, \)
\[
\int_{S^{n-1}} |p_k|^2 = O (k^{-N})
\]
as \( k \to \infty \), we can show that
\[
\int |\Delta_r^* f|^2 \leq c.
\]
Hence, \( f \in H^{2r} (S^{n-1}) \). Sobolev embedding implies \( f \) is smooth.

**Theorem 47** (Identity of Hecke). Suppose \( p \) is a solid spherical harmonic of degree \( k \) in \( \mathbb{R}^n \), then
\[
\hat{p} e^{-|x|^2/2} = (-i)^k p (y) e^{-|y|^2/2}.
\]
Proof. We have
\[
\hat{p}e^{-\frac{|x|^2}{2}}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} p(x) e^{-\frac{|x|^2}{2}} e^{-ix \cdot y} dx
\]
\[
= \frac{1}{(2\pi)^{n/2}} e^{-\frac{|y|^2}{2}} \int_{\mathbb{R}^n} p(x) e^{-\frac{|x+y|^2}{2}} dx
\]
\[
= \frac{1}{(2\pi)^{n/2}} e^{-\frac{|y|^2}{2}} \int_{\mathbb{R}^n} p(x - iy) e^{-\frac{|x|^2}{2}} dx
\]
\[
= q(y) e^{-\frac{|y|^2}{2}}
\]
where \( q \) is a polynomial. Replacing \( y \) by \( iy \), we also have
\[
q(iy) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} p(x + iy) e^{-\frac{|x|^2}{2}} dx = p(y)
\]
since \( p \) is harmonic. Hence
\[
q(y) = p(-iy) = (-i)^k p(y).
\]
\[\square\]

6.4. Higher Riesz Transforms. We consider for \( k \geq 1 \) the singular integral with the kernel \( \frac{p(x)}{|x|^{n+k}} \) where \( p \in \mathcal{H}_k \). When \( k = 1 \), we recover Riesz transform. Such singular integrals are hence called higher Riesz transforms.

The main theorem we want to establish is:

**Theorem 48.** Let \( p \in \mathcal{H}_k, k \geq 1 \). Then the multiplier corresponding to the transform
\[
Tf(y) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \hat{p}(x) \frac{p(x)}{|x|^{n+k}} f(y - x) dx
\]
is
\[
\gamma_k \frac{p(x)}{|x|^k}
\]
where
\[
\gamma_k = (-i)^k \frac{\pi \frac{n}{2} \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{k+n}{2} \right)}.
\]

**Remark 19.** When \( k = 1 \), we have
\[
\gamma_1 = -i \frac{\pi \frac{n}{2} \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3+n}{2} \right)} = -i \frac{\pi(n+1)/2}{\Gamma \left( \frac{n+1}{2} \right)}
\]
which recovers the multiplier for Riesz transform.

We first prove several lemmas.

**Lemma 13.** Let \( p \in \mathcal{H}_k, k \geq 0 \). Then
\[
pe^{-|x|^2/2} = \delta^{k-\frac{n}{2}} (-i)^k p(y) e^{-\frac{|y|^2}{2}}.
\]

**Proof.** Simple scaling. \[\square\]
Lemma 14. Let $p \in \mathcal{H}_k$, $k \geq 0$. For any $\alpha \in (0, n)$ and for any $\varphi \in \mathcal{S}$, we have
\[
\int_{\mathbb{R}^n} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx = \gamma_{k,\alpha} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{k+\alpha}} \varphi(x) \, dx
\]
where
\[
\gamma_{k,\alpha} = (-i)^k 2^{\alpha - \frac{n}{2}} \frac{\Gamma \left( \frac{k+n}{2} \right)}{\Gamma \left( \frac{k+n-\alpha}{2} \right)}.
\]
Proof. For any $\delta > 0$, we have
\[
\int_{\mathbb{R}^n} p(x) e^{-\delta |x|^2/2} \hat{\varphi}(x) \, dx = \delta^{-k/2} (-i)^k \int_{\mathbb{R}^n} p(x) e^{-\frac{|x|^2}{4\delta}} \varphi(x) \, dx.
\]
Multiplying both sides with $\delta^{\beta-1}$ with
\[\beta = \frac{k+n-\alpha}{2},\]
and integrate $\delta$ from 0 to $\infty$, we have
\[
\int_{\mathbb{R}^n} \left( \int_0^\infty \delta^{\beta-1} e^{-\delta |x|^2/2} d\delta \right) p(x) \hat{\varphi}(x) \, dx = (-i)^k \int_{\mathbb{R}^n} \left( \int_0^\infty \delta^{-\frac{k+n-1}{2} - \frac{1}{4\delta}} d\delta \right) p(x) \varphi(x) \, dx.
\]
Now
\[
\int_0^\infty \delta^{-\frac{k+n-1}{2} - \frac{1}{4\delta}} d\delta = \left( \frac{|x|^2}{2} \right)^{-\beta} \int_0^\infty u^{\frac{k+n}{2} - 1} e^{-u} du = 2^{\frac{k+n}{2}} \Gamma \left( \frac{k+n}{2} \right) \frac{1}{|x|^{k+n-\alpha}},
\]
and
\[
\int_0^\infty \delta^{-\frac{k+n-1}{2} - \frac{1}{4\delta}} d\delta = \left( \frac{|x|^2}{2} \right)^{-\frac{k+n}{2}} \int_0^\infty u^{\frac{k+n}{2} - 1} e^{-u} du = 2^{\frac{k+n}{2}} \Gamma \left( \frac{k+n}{2} \right) \frac{1}{|x|^{k+n-\alpha}}
\]
where we used $u = \frac{|x|^2}{4\delta}$ and $d\delta = -\frac{|x|^2}{4u^2} du$.
Hence,
\[
\int_{\mathbb{R}^n} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx = (-i)^k \frac{2^{\frac{k+n}{2}} \Gamma \left( \frac{k+n}{2} \right) \Gamma \left( \frac{k+n-\alpha}{2} \right)}{2^{\frac{k+n-1}{2}} \Gamma \left( \frac{k+n-\alpha}{2} \right)} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{k+\alpha}} \varphi(x) \, dx
\]
\[
= (-i)^k 2^{\alpha - \frac{n}{2}} \frac{\Gamma \left( \frac{k+n}{2} \right)}{\Gamma \left( \frac{k+n-\alpha}{2} \right)} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{k+\alpha}} \varphi(x) \, dx.
\]
\[
\square
\]

Lemma 15. Let $p \in \mathcal{H}_k$, $k \geq 1$. For any $\alpha \in (0, n)$ and for any $\varphi \in \mathcal{S}$, we have
\[
\lim_{\alpha \to 0^+} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{p(x)}{|x|^{n+k}} \hat{\varphi}(x) \, dx.
\]
Proof.
\[
\int_{\mathbb{R}^n} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx
\]
\[
= \int_{|x|\leq 1} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx + \int_{|x|\geq 1} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx
\]
\[
= \int_{|x|\leq 1} \frac{p(x)}{|x|^{n+k-\alpha}} [\hat{\varphi}(x) - \hat{\varphi}(0)] \, dx + \int_{|x|\geq 1} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx.
\]
Hence,
\[
\lim_{\alpha \to 0^+} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx
\]
\[
= \int_{|x|\leq 1} \frac{p(x)}{|x|^{n+k-\alpha}} [\hat{\varphi}(x) - \hat{\varphi}(0)] \, dx + \int_{|x|\geq 1} \frac{p(x)}{|x|^{n+k}} \hat{\varphi}(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0^+} \int_{|x|\geq \varepsilon} \frac{p(x)}{|x|^{n+k}} \hat{\varphi}(x) \, dx.
\]
\[\square\]

Now we can prove the main theorem.

**Proof of Theorem 48.** Let \( f \in S \). Then for any fixed \( x \), there exists \( \varphi \) such that
\[
f(x-y) = \hat{\varphi}(y)
\]
and we have
\[
\varphi(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x-z) e^{iy \cdot z} \, dz
\]
\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(u) e^{iy \cdot (x-u)} \, du
\]
\[
= \hat{f}(y) e^{ix \cdot y}.
\]
Hence
\[
\lim_{\varepsilon \to 0^+} \int_{|x|\geq \varepsilon} \frac{p(x)}{|x|^{n+k}} \hat{\varphi}(x) \, dx
\]
\[
= \lim_{\alpha \to 0^+} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{n+k-\alpha}} \hat{\varphi}(x) \, dx
\]
\[
= \lim_{\alpha \to 0^+} \gamma_{k,\alpha} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^{k+\alpha}} \varphi(x) \, dx
\]
\[
= \gamma_{k,0} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^k} \varphi(x) \, dx = \gamma_{k,0} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^k} e^{ix \cdot y} \hat{f}(x) \, dx
\]
where
\[
\gamma_{k,0} = (-i)^k 2^{-\frac{n}{2}} \frac{\Gamma \left( \frac{k}{2} \right)}{\Gamma \left( \frac{k+n}{2} \right)}.
\]
So we have
\[
Tf(y) = \lim_{\varepsilon \to 0^+} \int_{|x|\geq \varepsilon} \frac{p(x)}{|x|^{n+k}} f(y-x) \, dx = \gamma_{k,0} \int_{\mathbb{R}^n} \frac{p(x)}{|x|^k} e^{ix \cdot y} \hat{f}(x) \, dx.
\]
Hence,
\[ \hat{Tf}(x) = (2\pi)^{\frac{n}{2}} \gamma_{k,0} \frac{p(x)}{|x|^k} \hat{f}(x) = m(x) \hat{f}(x) \]
where
\[ m(x) = (2\pi)^{\frac{n}{2}} \gamma_{k,0} \frac{p(x)}{|x|^k} = \gamma_k \frac{p(x)}{|x|^k} . \]

Since \( \mathcal{S} \) is dense in \( L^2 \), the theorem is proved. \( \square \)

**Theorem 49.** Let \( c \) be a constant and \( \Omega \) be a homogeneous function on \( \mathbb{R}^n \) of degree 0, which is smooth on \( S^{n-1} \) and
\[ \int_{S^{n-1}} \Omega d\sigma = 0. \]

Define
\[ Tf(x) = cf(x) + \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y) dy. \]

Then there exists a homogeneous function \( m \) of degree 0 which is smooth on \( S^{n-1} \) such that
\[ \hat{Tf} = \hat{m} \hat{f}. \]

Moreover, the reverse is also true.

**Proof.** Write
\[ \Omega(x) = \sum_{k=1}^{\infty} p_k(x), \]
then on \( S^{n-1} \)
\[ m(x) = c + \sum_{k=1}^{\infty} \gamma_k p_k(x) \]
which is smooth whenever \( \Omega \) is smooth on \( S^{n-1} \). \( \square \)

7. Differentiability Properties in Terms of Function Spaces

7.1. Riesz Potential. Let \( f \in \mathcal{S} \). Recall
\[ (-\Delta)^{\frac{\beta}{2}} f(x) = |x|^\beta \hat{f}. \]

We can define for any \( \beta > -n \),
\[ \left( (-\Delta)^{\frac{\beta}{2}} f \right)^\vee(x) = |x|^{2\beta} \hat{f}. \]

Since \( |x|^{\beta} \hat{f} \in L^1(\mathbb{R}^n) \) for any \( \beta > -n \), \((-\Delta)^{\frac{\beta}{2}} f\) is well defined as inverse Fourier transform of integrable function.

For any \( \alpha \in (0, n) \), we define the Riesz potential of \( f \in \mathcal{S} \) by
\[ (I_{\alpha} f)(x) = (-\Delta)^{-\frac{\alpha}{2}} f. \]

Hence,
\[ \hat{I_{\alpha} f}(x) = |x|^{-\alpha} \hat{f} \]
and
\[ (I_{\alpha} f)(x) = \frac{1}{(2\pi)^{n/2}} \left( |x|^{-\alpha} \right)^\vee f. \]

Apparently, \( I_{\alpha} f \) is well defined as long as \( |x|^{-\alpha} \hat{f} \in L^1 \).
Proposition 12. For any \( f \in S \).

(a) 
\[
I_\alpha (I_\beta f) = I_{\alpha + \beta} f
\]
if \( \alpha > 0, \beta > 0 \) and \( \alpha + \beta < n \).

(b) If \( n \geq 3 \), then
\[
\Delta (I_\alpha f) = I_\alpha (\Delta f) = -I_{\alpha - 2} f
\]
holds for any \( \alpha \in (2, n) \). And when \( \alpha = 2 \), we have
\[
\Delta (I_2 f) = I_2 (\Delta f) = -f.
\]

Proof. (a) Since \( \alpha + \beta < n \)
\[
\hat{I}_{\alpha + \beta} f = |x|^{-\alpha - \beta} \hat{f} = |x|^{-\alpha} \hat{I}_\beta f = \hat{I}_\alpha (\hat{I}_\beta f).
\]
(b) For any \( \alpha \in (2, n) \),
\[
\Delta (I_\alpha f) = -|x|^2 \hat{I}_\alpha f = -|x|^{2-\alpha} \hat{f} = -\hat{I}_{\alpha - 2} f,
\]
\[
I_\alpha (\Delta f) = |x|^{-\alpha} \Delta f = -|x|^{2-\alpha} \hat{f} = -\hat{I}_{\alpha - 2} f.
\]
When \( \alpha = 2 \),
\[
\Delta (I_2 f) = I_2 (\Delta f) = -f
\]
follows similarly. \( \square \)

Now we calculate \( \left( |x|^{-\alpha} \right)^\vee \):

Lemma 16. Let \( \alpha \in (0, n) \). Then for any \( \phi \in S \),
\[
\int_{\mathbb{R}^n} |x|^{-n+\alpha} \hat{\phi}(x) \, dx = c_\alpha \int_{\mathbb{R}^n} |x|^{-\alpha} \phi(x) \, dx
\]
where
\[
\gamma_{0, \alpha} = 2^{n-\frac{n}{2}} \frac{\Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{n + \alpha}{2} \right)}.
\]
Hence,
\[
\frac{1}{(2\pi)^{n/2}} \left( |x|^{-\alpha} \right)^\vee = \frac{1}{\gamma_{0, \alpha}} |x|^{-n+\alpha}
\]
where
\[
\gamma_{\alpha} = 2^n \pi^{n/2} \frac{\Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{n + \alpha}{2} \right)}.
\]

Proof. Take \( k = 0 \) in Lemma 14, we have
\[
\int_{\mathbb{R}^n} |x|^{-n+\alpha} \hat{\phi}(x) \, dx = \gamma_{0, \alpha} \int_{\mathbb{R}^n} |x|^{-\alpha} \phi(x) \, dx
\]
where
\[
\gamma_{0, \alpha} = 2^{n-\frac{n}{2}} \frac{\Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{n + \alpha}{2} \right)}.
\]
Hence,
\[
\gamma_{0, \alpha} \int_{\mathbb{R}^n} |x|^{-\alpha} \phi(x) \, dx = \int_{\mathbb{R}^n} |x|^{-n+\alpha} \hat{\phi}(x) \, dx = \int_{\mathbb{R}^n} |x|^{-n+\alpha} (y) \phi(y) \, dy,
\]
i.e.,
\[
|\hat{x}|-n+\alpha = \gamma_{0, \alpha} |x|^{-\alpha}
\]
and
\[
\frac{1}{(2\pi)^{n/2}} \left( \frac{|x|^{-\alpha}}{\gamma_{0,\alpha} (2\pi)^{n/2}} \right) = \frac{1}{\gamma_{0,\alpha} (2\pi)^{n/2}} |x|^{-n+\alpha}.
\]

\[
\square
\]

**Corollary 9.** For any \( \alpha \in (0, n) \) and \( f \in \mathcal{S} \),
\[
(I_\alpha f)(x) = \frac{1}{\gamma_\alpha} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) \, dy
\]
where
\[
\gamma_\alpha = \frac{2^{\alpha/2} \pi^{n/2} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.
\]

We now ask for what pairs \( p \) and \( q \), we have
\[
\| I_\alpha f \|_q \leq A \| f \|_p.
\]
A necessary condition follows from a scaling argument. Let \((p, q)\) be such a pair. Since for any \( \delta > 0 \),
\[
I_\alpha (\tau_\delta f) = \delta^{-\alpha} \tau_\delta I_\alpha f
\]
and
\[
\| \tau_\delta f \|_p = \delta^{-n/p} \| f \|_p,
\]
we have
\[
\| I_\alpha (\tau_\delta f) \|_q = \delta^{-\alpha} \| \tau_\delta I_\alpha f \|_q = \delta^{-\alpha - \frac{\alpha}{p}} \| I_\alpha f \|_q \leq \delta^{-\alpha - \frac{\alpha}{p}} A \| f \|_p = \delta^{-\alpha - \frac{\alpha}{p} + \frac{n}{q}} A \| \tau_\delta f \|_p.
\]
Hence, we must have
\[
\frac{1}{q} = 1 - \frac{\alpha}{n}.
\]

**Theorem 50.** Let \( 0 < \alpha < n, \ 1 \leq p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and \( f \in L^p(\mathbb{R}^n) \).

(a) \( I_\alpha f \) defined in (7.1) converges absolutely for a.e. \( x \in \mathbb{R}^n \).

(b) If \( p > 1 \), then
\[
\| I_\alpha f \|_q \leq A_{p,q} \| f \|_p
\]
where \( A_{p,q} \) is a constant independent of \( f \).

(c) If \( p = 1 \), then for any \( t > 0 \),
\[
|\{ x : I_\alpha f(x) > t \}| \leq \left( \frac{A \| f \|_1}{t} \right)^q
\]
where \( A \) is a constant independent of \( f \).

**Proof.** (a) Fix \( \mu > 0 \). Let \( K = \frac{1}{\gamma_{\mu,\alpha} |x|^{n-\alpha}} \), \( K_1 = K \chi_{\{|x| \leq \mu\}} \) and \( K_2 = K - K_1 \). Then \( K_1 \in L^1 \) and \( K_2 \in L^{p'} \) where
\[
\frac{1}{p'} + \frac{1}{p} = 1.
\]
Moreover,
\[
\| K_1 \|_1 = \int_{|x| \leq \mu} \frac{1}{\gamma_{\mu,\alpha} |x|^{n-\alpha}} \, dx = \frac{n \omega_n}{\alpha \gamma_{\mu,\alpha}} \mu^\alpha \equiv c_1 \mu^\alpha,
\]

and
\[
\|K_2\|_{p'} = \left( \int_{|x| > \mu} \left( \frac{1}{\gamma_\alpha |x|^{n-\alpha}} \right)^{p'} dx \right)^{\frac{1}{p'}} = \frac{1}{\gamma_\alpha} \left( \frac{\omega_n q}{p'} \right)^{\frac{1}{p'}} \mu^{-\frac{n}{q}} \equiv c_2 \mu^{-\frac{n}{q}}.
\]
Hence for any \( f \in L^p \), \( K_1 * f \in L^p \), \( K_2 * f \in L^\infty \) and
\[
I_\alpha f = K * f = K_1 * f + K_2 * f.
\]
converges absolutely a.e. in \( \mathbb{R}^n \).

(b) (c) Assume \( \|f\|_p = 1 \). For each \( \mu > 0 \),
\[
\|K_1 * f\|_{p} \leq \|K_1\|_1 \|f\|_p = c_1 \mu^\alpha
\]
and
\[
\|K_2 * f\|_{\infty} \leq \|K_2\|_{p'} \|f\|_p = c_2 \mu^{-\frac{n}{q}}.
\]
For any \( t > 0 \), we have
\[
|\{x : |I_\alpha f(x)| > t\}| \leq \left| \left\{ x : (K_1 * f)(x) > \frac{t}{2} \right\} \right| + \left| \left\{ x : (K_2 * f)(x) > \frac{t}{2} \right\} \right|
\]
\[
\leq \frac{\|K_1 * f\|_{p}^p}{(t/2)^p} \leq \frac{1}{t^p} c_2 \mu^{-\frac{n}{q}}.
\]
if we choose
\[
\frac{t}{2} = c_2 \mu^{-\frac{n}{q}}.
\]
Hence,
\[
|\{x : |I_\alpha f(x)| > t\}| \leq c_1 \frac{1}{t^q},
\]
i.e., \( I_\alpha \) is of weak type \((p, q)\). In particular, (c) holds. Now the following Marcinkiewicz interpolation theorem implies (b).

**Theorem 51** (Marcinkiewicz). Assume \( 1 \leq p_i \leq q_i \leq \infty \), \( i = 1, 2 \) and \( p_1 < p_2 \), \( q_1 \neq q_2 \). Let \( T \) be a subadditive transformation defined on \( L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n) \). Suppose that \( T \) is of weak type \((p_i, q_i)\), \( i = 1, 2 \). Then for any \( \theta \in (0, 1) \) and
\[
\frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{(1 - \theta)}{q_1} + \frac{\theta}{q_2},
\]
\( T \) is of type \((p, q)\).

7.2. **The Sobolev spaces.** For any nonnegative integer \( k \), the Sobolev space \( W^{k,p}(\mathbb{R}^n) = L^p_{\text{loc}}(\mathbb{R}^n) \) is defined as the space of functions \( f \) with \( D^\alpha f \in L^p(\mathbb{R}^n) \) whenever \(|\alpha| \leq k \). \( W^{k,p}(\mathbb{R}^n) \) is a Banach space with the norm
\[
\|f\|_{W^{k,p}(\mathbb{R}^n)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_p^p \right)^{\frac{1}{p}}.
\]
An equivalent norm can be given by
\[
\sum_{|\alpha| \leq k} \|D^\alpha f\|_p.
\]
When $p = 2$, $H^k (\mathbb{R}^n) = W^{k,p} (\mathbb{R}^n)$ is a Hilbert space with inner product

$$(f,g) = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} D^\alpha f D^\alpha g dx.$$ 

**Proposition 13.** Let $1 \leq p < \infty$. Then $f \in W^{k,p} (\mathbb{R}^n)$ iff there exists a sequence $\{f_m\} \subset D(\mathbb{R}^n)$ s.t.

$$\lim_{m \to \infty} \|f - f_m\|_p = 0$$

and for each $|\alpha| \leq k$, $\{D^\alpha f_m\}$ converges in $L^p$ norm. In particular, $D(\mathbb{R}^n)$ is dense in $W^{k,p} (\mathbb{R}^n)$.

**Proof.** Let $\{\varphi_\epsilon\}$ be the standard mollifier. If $f \in W^{k,p} (\mathbb{R}^n)$, then $g_\epsilon = \varphi_\epsilon * f \in C^\infty (\mathbb{R}^n)$ and

$$\lim_{\epsilon \to 0} \|f - g_\epsilon\|_p = 0.$$

Since $D^\alpha g_\epsilon = \varphi_\epsilon * D^\alpha f$, for each $|\alpha| \leq k$

$$\lim_{\epsilon \to 0} \|D^\alpha f - D^\alpha g_\epsilon\|_p = 0.$$

Let $\eta \in D(\mathbb{R}^n)$ be a smooth cutoff function s.t. $\eta (x) = 1$ on $B_1 (0)$. For each $g \in C^\infty (\mathbb{R}^n) \cap W^{k,p} (\mathbb{R}^n)$, define $h_\delta (x) = g (x) \eta (\delta x)$, then

$$\lim_{\epsilon \to 0} \|h_\delta - g\|_p = 0.$$

The existence of $\{f_m\}$ follows from a Diagonal argument. The reverse is trivial. $\square$

**Lemma 17.** Suppose $f \in D(\mathbb{R}^n)$, then

$$f (x) = \frac{1}{n \omega_n} \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j} (x - y) \frac{y_j}{|y|} dy.$$ 

**Proof.** For any $\xi \in \mathbb{S}^{n-1}$, we have

$$f (x) = \int_0^\infty \xi \cdot \nabla f (x - \xi t) dt.$$ 

Integrating over $\xi \in \mathbb{S}^{n-1}$, we have

$$f (x) = \frac{1}{n \omega_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \xi \cdot \nabla f (x - \xi t) dt d\sigma(\xi)$$

$$= \frac{1}{n \omega_n} \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j} (x - y) \frac{y_j}{|y|} dy.$$ 

$\square$

**Lemma 18.** Suppose $f \in W^{1,1} (\mathbb{R}^n)$, then

$$\|f\|_{\mathbb{R}^n} \leq \left( \prod_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_1 \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_1.$$
Proof. We assume \( f \in D(\mathbb{R}^n) \). If \( n = 1 \), we have
\[
|f(x)| = \left| \int_{-\infty}^{x} f'(y) \, dy \right| \leq \|f'\|_1,
\]
hence
\[
\|f\|_\infty \leq \|f'\|_1.
\]
Suppose the inequality holds for \( n = k \), then for \( n = k + 1 \), we write \( x = (x', x_n) \), and
\[
\|f\|_\frac{\infty}{n-1} = \int_{\mathbb{R}^n} |f(x)|^\frac{n}{n-1} \, dx
\]
\[
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} |\frac{\partial f}{\partial x_n}| \, dx_n \right)^\frac{1}{n} |f(x)| \, dx' \right) \, dx_n
\]
\[
\leq \int_{\mathbb{R}} \left( \left( \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} |\frac{\partial f}{\partial x_n}| \, dx_n \right) \right)^\frac{1}{n-1} \left( \int_{\mathbb{R}^{n-1}} |f(x)|^\frac{n-1}{n-2} \, dx_n \right)^\frac{n-2}{n-1} \right) \, dx_n
\]
\[
\leq \|\frac{\partial f}{\partial x_n}\|_1^{\frac{n-1}{n}} \int_{\mathbb{R}} \left( \prod_{j=1}^{n-1} \|\frac{\partial f}{\partial x_j}\|_{L^1(\mathbb{R}^{n-1})} \right)^{\frac{n-1}{n}} \, dx_n
\]
\[
= \|\frac{\partial f}{\partial x_n}\|_1^{\frac{n-1}{n}} \left( \prod_{j=1}^{n-1} \int_{\mathbb{R}} \|\frac{\partial f}{\partial x_j}\|_{L^1(\mathbb{R}^{n-1})} \, dx_n \right)^{\frac{n-1}{n}}
\]
\[
= \left( \prod_{j=1}^{n} \|\frac{\partial f}{\partial x_j}\|_1 \right)^{\frac{1}{n}}.
\]
Since \( D(\mathbb{R}^n) \) is dense in \( W^{1,1}(\mathbb{R}^n) \), standard density argument implies that the inequality holds for \( f \in W^{1,1}(\mathbb{R}^n) \). \( \Box \)

The following theorem is essential for Sobolev spaces.

**Theorem 52.** Suppose \( k \) is a positive integer and \( \frac{1}{q} = \frac{1}{p} - \frac{k}{n} \).

(a) If \( 1 \leq p < \frac{n}{k} \), then \( W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \) and the natural inclusion is continuous.

(b) If \( p = \frac{n}{k} \geq 1 \), then \( W^{k,p}(\mathbb{R}^n) \subset L^r_{loc}(\mathbb{R}^n) \) for any \( 1 \leq r < \infty \).

(c) If \( p > \frac{n}{k} \) and \( p \geq 1 \), then \( W^{k,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \).

**Proof.** It suffices to prove the Theorem when \( k = 1 \) and the general case follows from an induction.

(a) Let \( f \in D(\mathbb{R}^n) \), then
\[
\sum_{j=1}^{n} R_j \left( \frac{\partial f}{\partial x_j} \right) = -i \frac{x_j}{|x|} \frac{\partial f}{\partial x_j} = |x| \hat{f}.
\]
Hence,
\[
f = I_1 \left( \sum_{j=1}^{n} R_j \left( \frac{\partial f}{\partial x_j} \right) \right)
\]
if \( n \geq 2 \). If \( 1 < p, q < \infty \), then \( n \geq 2 \), so part (i) when \( p > 1 \) follows from the properties of Riesz transform and Riesz potential. When \( p = 1 \), part (i) becomes

\[
\|f\|_{P_n} \leq \left( \prod_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_1 \right)^{-\frac{1}{n}}
\]

which is the lemma above.

(b) It follows from (a).

(c) Assume \( f \in D(R^n) \) with support in \( B_R(0) \), then for any \( x \in B_R(0) \)

\[
|f(x)| = \left| \frac{1}{n\omega_n} \sum_{j=1}^{n} \int_{R^n} \frac{\partial f}{\partial x_j} (x - y) \frac{y_j}{|y|^n} dy \right|
\]

\[
= \left| \frac{1}{n\omega_n} \sum_{j=1}^{n} \int_{|x-y| \leq R} \frac{\partial f}{\partial x_j} (x - y) \frac{y_j}{|y|^n} dy \right|
\]

\[
\leq \frac{1}{n\omega_n} \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_p \left\| \frac{y_j}{|y|^n} \right\|_{L^p(B_{2R}(0))}
\]

\[
\leq c \left( \frac{n}{\omega_n} \right) \left\| \frac{\partial f}{\partial x_j} \right\|_p
\]

since

\[
\left\| \frac{y_j}{|y|^n} \right\|_{L^p(B_{2R}(0))}^p \leq \int_{B_{2R}(0)} \frac{1}{|y|^{(n-1)p'}} dy
\]

\[
= n\omega_n \int_0^{2R} r^{-(n-1)p'+n-1} dr
\]

\[
= n\omega_n \frac{p-1}{p-n} (2R)^{\frac{p-n}{n}},
\]

here we used

\[-(n-1)p' + n = -\frac{(n-1)p}{p-1} + n = \frac{p-n}{p-1} > 0.\]

Density argument yields the conclusion for \( f \in W^{1,p}(R^n) \). \( \square \)

7.3. Bessel Potential and Bessel Spaces. Bessel potential \( J_\alpha \) is defined by

\[ J_\alpha = (I - \Delta)^{-\frac{\alpha}{2}}. \]

Hence, for any \( f \in S \),

\[ \widehat{J_\alpha f} = (1 + |x|^2)^{-\frac{\alpha}{2}} \hat{f} \]

and

\[ J_\alpha f = G_\alpha * f \]

where

\[ G_\alpha (x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left[ (1 + |x|^2)^{-\frac{\alpha}{2}} \right] \]

\[ = \frac{1}{(2\pi)^{n}} \int_{R^n} \left[ (1 + |y|^2)^{-\frac{\alpha}{2}} \right] e^{ix \cdot y} dy. \]
If $\alpha > n$, then $(1 + |x|^2)^{-\frac{\alpha}{2}} \in L^1$, and hence

$$G_\alpha (x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(1 + |y|^2\right)^{-\frac{\alpha}{2}} e^{ix\cdot y} dy.$$

Applying the identity

$$\left(1 + |x|^2\right)^{-\frac{\alpha}{2}} = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t(1+|x|^2)} t^{\frac{\alpha}{2}-1} dt,$$

we have

$$G_\alpha (x) = \frac{1}{(2\pi)^n \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \left(\int_0^\infty e^{-t(1+|y|^2)} t^{\frac{\alpha}{2}-1} dt\right) e^{ix\cdot y} dy$$

$$= \frac{1}{(2\pi)^n \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} e^{-t|y|^2 + i x\cdot y} dy\right) dt$$

$$= \frac{\pi^{\frac{\alpha}{2}}}{(2\pi)^n \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt$$

$$= \frac{1}{(4\pi)^\frac{n}{2} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt.$$

Now we claim that the above expression holds whenever $\alpha > 0$.

**Proposition 14.** For any $\alpha > 0$, define

$$G_\alpha (x) = \frac{1}{(4\pi)^\frac{n}{2} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt.$$

Then $G_\alpha \in L^1 (\mathbb{R}^n)$ and

$$\hat{G}_\alpha (y) = \frac{1}{(2\pi)^\frac{n}{2}} \left(1 + |y|^2\right)^{-\frac{\alpha}{2}}.$$

**Proof.** Since

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx = \int_{\mathbb{R}^n} e^{-\frac{|u|^2}{4t}} (4t)^\frac{n}{2} du = (4\pi t)^\frac{n}{2},$$

we have

$$\int_{\mathbb{R}^n} G_\alpha (x) dx = \frac{1}{(4\pi)^\frac{n}{2} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_{\mathbb{R}^n} e^{-t} t^{\frac{\alpha}{2}-1} dt\right) dx$$

$$= \frac{1}{(4\pi)^\frac{n}{2} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_{\mathbb{R}^n} e^{-t} t^{\frac{\alpha}{2}-1} dt\right) dx$$

$$= \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt = 1.$$
Now,

\[ (2\pi)^{\frac{n}{2}} \widehat{G_\alpha}(y) = \int_{\mathbb{R}^n} G_\alpha(x) e^{ix \cdot y} dx \]

\[ = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left( \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} e^{ix \cdot y} dx \right) e^{-t \frac{n+n}{2}-1} dt \]

\[ = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left( \int_{\mathbb{R}^n} e^{-\frac{1}{4t}|x-2iy|^2-1} dx \right) e^{-t \frac{n+n}{2}-1} dt \]

\[ = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty (4\pi t)^{\frac{n}{2}} e^{-t(1+|y|^2)} t \frac{n+n}{2}-1 dt \]

\[ = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-t(1+|y|^2)} t \frac{n+n}{2}-1 dt = \left(1 + |y|^2\right)^{-\frac{n}{2}}. \]

□

**Proposition 15.** (a) If \(0 < \alpha < n\),

\[ G_\alpha(x) = \frac{|x|^{-n+\alpha}}{\gamma(\alpha)} + o\left(|x|^{-n+\alpha}\right) \quad \text{as} \quad |x| \to 0. \]

(b) For any \(\alpha > 0\),

\[ G_\alpha(x) = O\left(e^{-\frac{1}{2}|x|}\right) \quad \text{as} \quad |x| \to \infty. \]

**Proof.** (a). Let

\[ u = \frac{|x|^2}{4t}, du = -\frac{|x|^2}{4u^2} du. \]

Hence,

\[ \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-\frac{|x|^2}{4u}} t \frac{n+n}{2}-1 dt \]

\[ = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-u} \left(\frac{|x|^2}{4u}\right) \frac{n+n}{2}-1 \frac{|x|^2}{4u^2} du \]

\[ = \frac{\frac{\alpha}{2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-u} u \frac{n+n}{2}-1 du \]

\[ = \frac{|x|^{-n+\alpha}}{\gamma(\alpha)} \]

where

\[ \gamma(\alpha) = \frac{2^{\alpha} \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \]
So we have
\[
\frac{1}{\gamma(\alpha)} - |x|^{-\alpha} G_\alpha(x) = \frac{|x|^{-\alpha}}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left(1 - e^{-t}\right) e^{-\frac{|x|^2}{4t} \frac{\alpha-n}{2}} t^{-\frac{\alpha-n}{2}-1} dt
\]
\[
= \frac{|x|^{-\alpha}}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left(1 - e^{-\frac{|x|^2}{4u}}\right) e^{-u \left(\frac{|x|^2}{4u}\right) \frac{\alpha-n}{2}} \frac{|x|^2}{4u^2} du
\]
\[
= \frac{1}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \left(1 - e^{-\frac{|x|^2}{4u}}\right) e^{-u \frac{\alpha-n}{2}} du \to 0 \text{ as } x \to 0.
\]

(b) When $|x| \geq 1$
\[
G_\alpha(x) = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-t - \frac{|x|^2}{4t} \frac{\alpha-n}{2}} t^{-\frac{\alpha-n}{2}-1} dt
\]
\[
\leq \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-\frac{1}{4t} - \frac{1}{4t} - \frac{|x|^2}{4t} \frac{\alpha-n}{2}} t^{-\frac{\alpha-n}{2}-1} dt
\]
\[
\leq \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-\frac{1}{4t} - \frac{1}{4t} - \frac{|x|^2}{4t} \frac{\alpha-n}{2}} t^{-\frac{\alpha-n}{2}-1} dt
\]
\[
\leq ce^{-\frac{|x|^2}{4}}.
\]
\[
\square
\]

For any $\alpha > 0$, since $G_\alpha \in L^1(\mathbb{R}^n)$, for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, the Bessel potential of $f$,
\[
J_\alpha f = G_\alpha \ast f \in L^p(\mathbb{R}^n)
\]
is well defined. We also define $J_0 f = f$ for any $f \in L^p(\mathbb{R}^n)$.

**Proposition 16.** For any $\alpha, \beta \geq 0$,
\[
J_\alpha J_\beta = J_{\alpha+\beta}.
\]

**Lemma 19.** Let $\alpha \geq 0$.
(a) $J_\alpha$ is a bijection from $S$ onto $S$.
(b) Let $1 \leq p \leq \infty$ and $g_1, g_2 \in L^p(\mathbb{R}^n)$. Then $J_\alpha (g_1) = J_\alpha (g_2)$ implies $g_1 = g_2$.

**Proof.** (a) Since Fourier transform is a bijection from $S$ onto $S$, it follows from the definition that for any $f \in S$,
\[
\hat{J_\alpha f} = \left(1 + |\hat{f}|^2\right)^{-\frac{n}{2}} \hat{f}.
\]
(b) Let $g = g_1 - g_2$. $J_\alpha (g) = 0$ implies that for any $\varphi \in S$,
\[
0 = \int_{\mathbb{R}^n} J_\alpha (g) \varphi = \int_{\mathbb{R}^n} g J_\alpha (\varphi).
\]
Since $J_\alpha (\varphi)$ can be any function in $S$, $g = 0$. \qed

**Definition 13.** The potential space $\mathcal{L}_\alpha^p(\mathbb{R}^n)$ is defined as
\[
\mathcal{L}_\alpha^p = J_\alpha (L^p(\mathbb{R}^n)).
\]
The $L^p_{\alpha}$-norm of $f$ is defined by

$$\|f\|_{p,\alpha} = \|g\|_p \text{ if } f = J_\alpha (g).$$

**Remark 20.** $\|\cdot\|_{p,\alpha}$ is indeed well defined since $J_\alpha (g_1) = J_\alpha (g_2)$ implies $g_1 = g_2$. Obviously $L^p_{\alpha}$ is a Banach space.

**Lemma 20.** Let $\alpha > 0$.

(i) There exists a finite measure $\mu_\alpha$ on $\mathbb{R}^n$ s.t. its Fourier transform is given by

$$\hat{\mu}_\alpha (x) = \frac{|x|^\alpha}{(1 + |x|^2)^{\frac{\alpha}{2}}}.$$

(ii) There exists a finite measure $\nu_\alpha$ on $\mathbb{R}^n$ s.t.

$$\left(1 + |x|^2\right)^{\frac{\alpha}{2}} = \hat{\nu}_\alpha (x) \left(1 + |x|^\alpha\right).$$

**Proof.** (a)

$$\frac{|x|^\alpha}{\left(1 + |x|^2\right)^{\frac{\alpha}{2}}} = \left(1 - \frac{1}{1 + |x|^2}\right)^{\frac{\alpha}{2}}$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\alpha}{k}\right) \left(1 + |x|^2\right)^{-k}$$

$$= (2\pi)^{\frac{n}{2}} \left[\delta_0 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\alpha}{k}\right) G_{2k} (x)\right].$$

Here

$$\mu_\alpha = (2\pi)^{\frac{n}{2}} \left[\delta_0 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\alpha}{k}\right) G_{2k} (x)\right]$$

is a finite measure.

(b) We need Weiner’s theorem: If $\varphi \in L^1 (\mathbb{R}^n)$ satisfies $\hat{\varphi} + 1$ is nowhere zero, then there exists $\psi \in L^1 (\mathbb{R}^n)$ s.t.

$$(\hat{\varphi} + 1)^{-1} = \hat{\psi} + 1.$$

Let

$$\varphi = (2\pi)^{\frac{n}{2}} \left[\sum_{k=1}^{\infty} (-1)^k \left(\frac{\alpha}{k}\right) G_{2k} (x) + G_\alpha (x)\right],$$

then

$$\hat{\varphi} + 1 = \frac{|x|^\alpha + 1}{\left(1 + |x|^2\right)^{\frac{\alpha}{2}}}$$

never vanishes, hence, there exists $\psi \in L^1 (\mathbb{R}^n)$ s.t.

$$\hat{\psi} + 1 = \frac{\left(1 + |x|^2\right)^{\frac{\alpha}{2}}}{|x|^\alpha + 1},$$
i.e.,

\[ (1 + |x|^2)^{\frac{\alpha}{2}} = \hat{\psi} + 1 + |x|^{\alpha} (\hat{\psi} + 1). \]

Thus we can take \( \nu = \psi + (2\pi)^{\frac{n}{2}} \delta_0. \)

**Lemma 21.** Suppose \( 1 < p < \infty \), and \( \alpha \geq 1 \). Then \( f \in \mathcal{L}_p^\alpha (\mathbb{R}^n) \) iff \( f \in \mathcal{L}_\alpha^{-1}^p (\mathbb{R}^n) \) and for each \( j \), \( \frac{\partial f}{\partial x_j} \in \mathcal{L}_\alpha^{-1}^p (\mathbb{R}^n) \). Moreover, the two norms \( \|f\|_{p, \alpha} \) and

\[ \|f\|_{p, \alpha} = \|f\|_{p, \alpha - 1} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{p, \alpha - 1} \]

are equivalent.

**Proof.** 1. If \( f \in \mathcal{L}_\alpha^p (\mathbb{R}^n) \), then there exists \( g \in L^p (\mathbb{R}^n) \), \( f = J_\alpha (g) \). Since

\[ f = J_\alpha (g) = J_{\alpha - 1} (J_1 g) \]

and \( J_1 g \in L^p (\mathbb{R}^n) \), we have \( f \in \mathcal{L}_\alpha^{-1}^p (\mathbb{R}^n) \) with

\[ \|f\|_{p, \alpha} = \|J_1 g\|_p \leq \|g\|_p = \|f\|_{p, \alpha}. \]

Now for each \( j \),

\[ \frac{\partial f}{\partial x_j} = \frac{\partial J_\alpha}{\partial x_j} g = -ix_j \hat{\psi} + (1 + |x|^2)^{\frac{\alpha}{2} - \frac{2}{2}} \hat{g} + \left( \hat{\mu_1} \cdot \hat{R}_j g \right) \]

Hence,

\[ \frac{\partial f}{\partial x_j} = J_{\alpha - 1} \left( (2\pi)^{-\frac{n}{2}} \hat{\mu_1} \cdot \hat{R}_j g \right) \in \mathcal{L}_\alpha^{-1} (\mathbb{R}^n). \]

Moreover,

\[ \left\| \frac{\partial f}{\partial x_j} \right\|_{p, \alpha - 1} = \left\| (2\pi)^{-\frac{n}{2}} \hat{\mu_1} \cdot \hat{R}_j g \right\|_p \leq c \|R_j g\|_p \leq C \|g\|_p. \]

2. If \( f \in \mathcal{L}_\alpha^{-1}^p (\mathbb{R}^n) \) and for each \( j \), \( \frac{\partial f}{\partial x_j} \in \mathcal{L}_\alpha^{-1}^p (\mathbb{R}^n) \). Then there exists \( g \) and \( g_j \) in \( L^p (\mathbb{R}^n) \), such that

\[ f = J_\alpha^{-1} (g) \] and for each \( j \),

\[ \frac{\partial f}{\partial x_j} = J_\alpha^{-1} (g_j). \]

Since \( \hat{f} = (1 + |x|^2)^{-\frac{\alpha+1}{2}} \hat{g} \), we have

\[ \frac{\partial f}{\partial x_j} = -ix_j \hat{f} \left( 1 + |x|^2 \right)^{-\frac{\alpha+1}{2}} (-ix_j) \hat{g} = \left( 1 + |x|^2 \right)^{-\frac{\alpha+1}{2}} \frac{\partial g}{\partial x_j}. \]
However,
\[
\frac{\partial f}{\partial x_j} = \left(1 + |x|^2\right)^{-\frac{n+1}{2}} \hat{g}_j = \left(1 + |x|^2\right)^{-\frac{n+1}{2}} \frac{\partial g}{\partial x_j}.
\]
Hence, we have
\[
\frac{\partial g}{\partial x_j} = g_j.
\]

Now, the following formal calculation can be justified,
\[
\hat{f} = \left(1 + |x|^2\right)^{-\frac{n+1}{2}} \hat{g} = \left(1 + |x|^2\right)^{-\frac{n}{2}} \left(1 + |x|^2\right)^{\frac{1}{2}} \hat{g} = \left(1 + |x|^2\right)^{-\frac{n}{2}} \nu_\alpha(x) (1 + |x|) \hat{g} = \left(1 + |x|^2\right)^{-\frac{n}{2}} \left(2\pi\right)^{-\frac{n}{2}} \nu_\alpha \ast \left(\hat{g} + R_j \left(\frac{\partial g}{\partial x_j}\right)\right).
\]

Hence, \( f \in L_k^p (\mathbb{R}^n) \) and
\[
\|f\|_{p,\alpha} = \left\|\left(2\pi\right)^{-\frac{n}{2}} \nu_\alpha \ast \left(\hat{g} + R_j \left(\frac{\partial g}{\partial x_j}\right)\right)\right\|_p
\leq c \left\|\hat{g} + R_j \left(\frac{\partial g}{\partial x_j}\right)\right\|_p
\leq c \left(\|\hat{g}\|_p + \sum \left\|\frac{\partial g}{\partial x_j}\right\|_p\right).
\]

\[\square\]

**Theorem 53.** Suppose \( k \) is a nonnegative integer and \( p \in (1, \infty) \). Then
\[
L_k^p (\mathbb{R}^n) = W^{k,p} (\mathbb{R}^n).
\]

**Proof.** By induction. \[\square\]

Suppose that \( f \in L^p (\mathbb{R}^n) \). We define the \( L^p \) modulus of continuity:
\[
\omega_p (t) = \|f (x + t) - f (x)\|_p.
\]
Then
\[
\lim_{|t| \to 0} \omega_p (t) = 0.
\]

Let’s first recall the Minkowski integral inequality.

**Theorem 54.** Let \((X, d\mu)\) and \((Y, d\nu)\) be two \( \sigma \)-finite measure spaces and \( 1 \leq p < \infty \). Then for any measurable function \( F(x, y) \) on \( X \times Y \),
\[
\left( \int_Y \left( \int_X |F(x, y)|^p \, d\mu_x \right)^\frac{1}{p} \, d\nu_y \right)^\frac{1}{p} \leq \int_X \left( \int_Y |F(x, y)|^p \, d\nu_y \right)^\frac{1}{p} \, d\mu_x.
\]

If \((X, d\mu)\) is the counting measure for \( \{1, 2\} \) and \((Y, d\nu)\) is the Lebesgue measure on \( \mathbb{R}^n \). We deduce
\[
\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p.
\]


**Proposition 17.** Suppose $1 < p < \infty$. Then $f \in \mathcal{L}^p_1(\mathbb{R}^n)$ iff $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $\omega_p(t) = O(|t|)$ as $|t| \to 0$.

**Proof.** Suppose $f \in \mathcal{D}(\mathbb{R}^n)$. Let $t = |t|t'$.

$$f(x + t) - f(x) = \int_0^{|t|} (\nabla f, t') (x + st') \, ds.$$ 

Hence, Minkowski’s inequality implies

$$\|f(x + t) - f(x)\|_p \leq |t| \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_p.$$ 

Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{L}^p_1(\mathbb{R}^n)$, for any $f \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\|f(x + t) - f(x)\|_p \leq |t| \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_p$$

and hence $\omega_p(t) = O(|t|)$ as $|t| \to 0$.

On the other hand, suppose $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $\omega_p(t) = O(|t|)$ as $|t| \to 0$. Let $e_j$ be the unit vector in $j$-th direction. Since

$$\left\| \frac{f(x + se_j) - f(x)}{s} \right\|_p = O(1),$$

there exists a subsequence, $s_k \to 0$, s.t.,

$$\frac{f(x + s_k e_j) - f(x)}{s_k} \to f_j$$ weakly in $\mathcal{L}^p(\mathbb{R}^n)$ as $k \to \infty$.

Since for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{f(x + s_k e_j) - f(x)}{s_k} \varphi(x) \, dx = - \int_{\mathbb{R}^n} f(x) \frac{\varphi(x) - \varphi(x - s_k e_j)}{s_k} \, dx,$$

we have, letting $k \to \infty$,

$$\int_{\mathbb{R}^n} f_j \varphi(x) \, dx = - \int_{\mathbb{R}^n} f(x) \frac{\partial \varphi}{\partial x_j} \, dx.$$ 

Hence,

$$\frac{\partial f}{\partial x_j} = f_j \in \mathcal{L}^p(\mathbb{R}^n)$$

and $f \in \mathcal{L}^p_1(\mathbb{R}^n)$. \qed

### 8. Extensions and Restrictions

**8.1. Decomposition of open sets into cubes.** Let’s recall the following theorem we proved.

**Theorem 55.** Let $F$ be a nonempty closed set in $\mathbb{R}^n$ s.t. $F \neq \mathbb{R}^n$. Then its complement $\Omega = F^c$ can be written as a union of cubes

$$\Omega = \bigcup_{k=1}^{\infty} Q_k,$$

whose interiors are disjoint and for each $k$,

$$\text{diam } Q_k \leq \text{dist } (Q_k, F) \leq 4 \text{ diam } Q_k.$$
Let \( \mathcal{F} \) be the collection \( \{Q_k\}_{k=1}^{\infty} \). We say that two cubes \( Q_1, Q_2 \) in \( \mathcal{F} \) touch if \( Q_1 \cap Q_2 \neq \emptyset \).

**Proposition 18.** Suppose two cubes \( Q_1, Q_2 \in \mathcal{F} \) touch. Then

\[
\frac{1}{4} \text{diam} Q_1 \leq \text{diam} (Q_1) \leq 4 \text{diam} Q_2.
\]

**Proof.** Since \( Q_1 \cap Q_2 \neq \emptyset \),

\[
\text{diam} Q_2 \leq \text{dist} (Q_2, \mathcal{F}) \leq \text{dist} (Q_1, \mathcal{F}) + \text{diam} Q_1 \leq 5 \text{diam} Q_1.
\]

Since \( \text{diam} Q_2 = 2^k \text{diam} Q_1 \) for some integer \( k \), we deduce

\[
\text{diam} Q_2 \leq 4 \text{diam} Q_1.
\]

\[\square\]

**Proposition 19.** Suppose \( Q \in \mathcal{F} \). Then there are at most \( N = 12^n \) cubes in \( \mathcal{F} \) which touch \( Q \).

Let \( \varepsilon \in (0, \frac{1}{4}) \) be fixed. For any \( Q_k \in \mathcal{F} \) with center \( x_k \), we define

\[
Q_k^* = (1 + \varepsilon) (Q_k - x_k) + x_k.
\]

**Proposition 20.** Each point of \( \Omega \) is contained in at most \( N \) of the cubes \( Q_k^* \).

**Proof.** Let \( Q \) and \( Q_k \) be two cubes in \( \mathcal{F} \). We consider the union of \( Q_k \) with all cubes in \( \mathcal{F} \) which touches \( Q_k \), then this union contains \( Q_k^* \) as its interior because of the proposition 18. Hence, \( Q_k^* \cap Q \neq \emptyset \) iff \( Q_k \cap Q \neq \emptyset \). \[\square\]

Let \( Q_0 \) be the unit cube centered at the origin. Fix \( \varphi \in \mathcal{D}(\mathbb{R}^n) \), s.t. \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) on \( Q_0 \) and \( \varphi = 0 \) on \( \mathbb{R}^n \setminus Q_0 \). Let

\[
\varphi_k (x) = \varphi \left( \frac{x - x_k}{l_k} \right)
\]

where \( x_k \) is the center of \( Q_k \) and \( l_k \) is the side length of \( Q_k \).

We now define

\[
\varphi_k^* (x) = \frac{\varphi_k (x)}{\sum_{k=1}^{\infty} \varphi_k (x)}, \quad x \in \Omega.
\]

Then we have

\[
\sum_{k=1}^{\infty} \varphi_k^* (x) \equiv 1 \text{ for } x \in \Omega = \mathcal{F}^c.
\]

### 8.2. Extension theorems of Whitney type

We first consider the regularized distance. Let \( F \) be a nonempty closed set in \( \mathbb{R}^n \) s.t. \( F \) is not the whole space. Let \( \delta (x) \) be the distance of \( x \) from \( F \). Then \( \delta \) is a Lipschitz function.

**Theorem 56.** There exists a function \( \Delta x = \Delta (x, F) \) defined in \( F^c \) s.t.

(a) \( c_1 \delta (x) \leq \Delta x \leq c_2 \delta (x), \quad x \in F^c; \)

(b) \( \Delta x \) is smooth in \( F^c \) and

\[
\left| \frac{\partial^\alpha}{\partial x^{\alpha}} \Delta (x) \right| \leq B_\alpha (\delta (x))^{1-|\alpha|}
\]

where \( B_\alpha, c_1, c_2 \) are constants independent of \( F \).
Proof. Let
\[ \Delta x = \sum_{k=1}^{\infty} (\text{diam } Q_k) \varphi_k (x). \]
If \( x \in Q_k \), then
\[ \delta (x) = \text{dist} (x, F) \leq \text{dist} (Q_k, F) + \text{diam } Q_k \leq 5 \text{diam } Q_k, \]
and hence
\[ \Delta x = \sum_{k=1}^{\infty} (\text{diam } Q_k) \varphi_k (x) \geq (\text{diam } Q_k) \varphi_k (x) = \text{diam } Q_k \geq \frac{1}{5} \delta (x). \]
Also if \( x \in Q_k^* \), then
\[ \delta (x) \geq \text{dist} (Q_k, F) - \frac{1}{4} \text{diam } Q_k \geq \frac{3}{4} \text{diam } Q_k. \]
Since any \( x \in F^c \) lies in at most \( N \) of the \( Q_k^* \), we have
\[ \Delta x \leq \sum_{x \in Q_k^*} \text{diam } Q_k \leq \frac{4N}{3} \delta (x). \]
For any \( \alpha, |\alpha| \geq 1 \), there exists \( A_\alpha \) such that
\[ |D^\alpha \varphi_k (x)| \leq A_\alpha (\text{diam } Q_k)^{-|\alpha|} \]
\( \Delta x \) is smooth in \( F^c \) and for any \( x \in F^c \),
\[ |D^\alpha \Delta (x)| \leq \sum_{x \in Q_k^*} \text{diam } Q_k |D^\alpha \varphi_k (x)| \]
\[ \leq \sum_{x \in Q_k^*} A_\alpha (\text{diam } Q_k)^{1-|\alpha|} \]
\[ \leq NA_\alpha \left( \frac{1}{6} \delta (x) \right)^{1-|\alpha|} \]
where we used the fact that if \( x \in Q_k^* \), then
\[ \delta (x) = \text{dist} (x, F) \leq \text{dist} (Q_k, F) + 2 \text{diam } Q_k \leq 6 \text{ diam } Q_k. \]

Proposition 21. \( E_0 (f) \) is continuous on \( \mathbb{R}^n \) and smooth in \( F^c \).
First, we observe that for any \( y \in F \), there exists a nbhd of \( x \) such that the summation is finite. Now we need to show the continuity of \( \mathcal{E}_0 ( f ) \) for each \( y \in F \). If \( x \in Q^*_k \), then
\[
| y - p_k | \leq | y - x | + | x - p_k | \leq c | y - x |.
\]
For each \( \varepsilon > 0 \), there exists \( \delta, s.t. \)
\[
| f ( z ) - f ( y ) | < \varepsilon
\]
whenever \( | z - y | < \delta \). Now for any \( x \in F \), s.t. \( | y - x | < \frac{\delta}{\varepsilon} \),
\[
| \mathcal{E}_0 ( f ) ( x ) - f ( y ) | \leq \sum_{k=1}^{\infty} | f ( p_k ) - f ( y ) | \varphi^*_k ( x )
\]
\[
= \sum_{x \in Q^*_k} | f ( p_k ) - f ( y ) | \varphi^*_k ( x ) < \varepsilon \sum_{x \in Q^*_k} \varphi^*_k ( x ) \leq \varepsilon.
\]
Hence \( \mathcal{E}_0 ( f ) \) is continuous at \( y \).

For any \( 0 < \gamma \leq 1 \), we define
\[
\text{Lip} ( \gamma, \mathbb{R}^n ) = \{ f : | f ( x ) | \leq M, | f ( x ) - f ( y ) | \leq M | x - y |^\gamma, x, y \in \mathbb{R}^n \}.
\]
Then \( \text{Lip} ( \gamma, \mathbb{R}^n ) \) is a Banach space with norm
\[
\| f \|_{C^{\gamma}} = \| f \|_\infty + [ f ]_\gamma = \| f \|_\infty + \inf_{x, y \in \mathbb{R}^n} \frac{| f ( x ) - f ( y ) |}{| x - y |^\gamma}.
\]
We remark here that if \( \gamma > 1 \), then \( \text{Lip} ( \gamma, \mathbb{R}^n ) \) defined as above contains only constant functions.

If \( F \subset \mathbb{R}^n \) is a nonempty closed set, we can define the Banach space \( \text{Lip} ( \gamma, F ) \) in a similar way.

**Theorem 57.** The linear extension operator \( \mathcal{E}_0 \) maps \( \text{Lip} ( \gamma, F ) \) continuously into \( \text{Lip} ( \gamma, \mathbb{R}^n ) \) if \( 0 < \gamma \leq 1 \). The norm of this mapping has a bound independent of the closed set \( F \).

**Proof.** First, we observe that for any \( x \in F^c \),
\[
| D^\alpha \varphi^*_k ( x ) | \leq A'_\alpha ( \text{diam } Q_k )^{-|\alpha|}.
\]
Now we claim for any \( x \in F^c \),
\[
\left| \frac{\partial}{\partial x_i} \mathcal{E}_0 ( f ) ( x ) \right| \leq c [ f ]_\gamma ( \delta ( x ) )^{\gamma-1}.
\]
In fact, for any \( x \in F^c \), choose \( y \in F \) s.t. \( \delta ( x ) = | x - y | \), we have
\[
\left| \frac{\partial}{\partial x_i} \mathcal{E}_0 ( f ) ( x ) \right| = \sum_{k=1}^{\infty} | f ( p_k ) - f ( y ) | \frac{\partial \varphi^*_k ( x )}{\partial x_i}
\]
\[
\leq \sum_{k=1}^{\infty} | f ( p_k ) - f ( y ) | \left| \frac{\partial \varphi^*_k ( x )}{\partial x_i} \right|
\]
\[
\leq c [ f ]_\gamma ( \delta ( x ) )^{\gamma-1}
\]
where \( c \) only depends on \( n \).
Now for any $x \in F^c$, $y \in F$,

$$|\mathcal{E}_0(f)(x) - \mathcal{E}_0(f)(y)| = \left| \sum_{k=1}^{\infty} (f(p_k) - f(y)) \varphi_k^*(x) \right|$$

$$\leq c \sum_{x \in Q_k} |f(p_k) - f(y)|$$

$$\leq c [f]_\gamma \sum_{x \in Q_k} |p_k - y|^\gamma \leq c [f]_\gamma |x - y|^\gamma.$$

For any $x, y \in F^c$, let $L$ be the line segment from $x$ to $y$.

**Case I:** $|x - y| \leq \text{dist}(L, F)$. Then

$$|\mathcal{E}_0(f)(x) - \mathcal{E}_0(f)(y)| \leq |x - y| \sup_{x' \in L} |\nabla \mathcal{E}_0(f)(x')|$$

$$\leq c [f]_\gamma |x - y| (\delta(x))^{\gamma - 1}$$

$$\leq c [f]_\gamma |x - y|^\gamma.$$

**Case II:** $|x - y| > \text{dist}(L, F)$. Then there exists $x' \in L$ and $y' \in F$ s.t.

$$|x' - y'| < |x - y|,$$

hence

$$|x - y'| < 2 |x - y| \text{ and } |y - y'| < 2 |x - y|.$$

So we have

$$|\mathcal{E}_0(f)(x) - \mathcal{E}_0(f)(y)|$$

$$\leq |\mathcal{E}_0(f)(x) - \mathcal{E}_0(f)(y') - |\mathcal{E}_0(f)(y) - \mathcal{E}_0(f)(y')|$$

$$\leq c [f]_\gamma |x - y'|^{\gamma} + c [f]_\gamma |y - y'|^{\gamma} \leq c [f]_\gamma |x - y|^\gamma.$$

More generally, let $\omega(\delta)$ be a modulus of continuity satisfying

(i) $\frac{\omega(\delta)}{\delta}$ is increasing as $\delta \to 0$;

(ii) $\omega(2\delta) \leq c \omega(\delta)$ for any $\delta > 0$.

We define $\text{Lip}(\omega, F)$ to define the collection of continuous functions $f$ such that

$$\|f\|_{\text{C}^\omega} = \|f\|_{\infty} + [f]_\omega = \|f\|_{\infty} + \inf_{x, y \in \mathbb{R}^n \atop x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)} < \infty.$$

Then $\text{Lip}(\omega, F)$ is a Banach space.

**Corollary 10.** The linear extension operator $\mathcal{E}_0$ maps $\text{Lip}(\omega, F)$ continuously into $\text{Lip}(\omega, \mathbb{R}^n)$.

Let $\gamma > 1$ and $k$ be an integer s.t. $k < \gamma \leq k + 1$. We say that $f \in \text{Lip}(\gamma, F)$ if $f$ is defined on $F$ and there exists $f^{(\alpha)}, |\alpha| \leq k$, such that, for any $0 \leq |\alpha| \leq k$, if

$$f^{(\alpha)}(x) = \sum_{|\alpha + \beta| \leq k} \frac{f^{(\alpha + \beta)}(y)}{\beta!} (x - y)^\beta + R_\alpha(x, y),$$

then

$$|f^{(\alpha)}(x)| \leq M \text{ and } |R_\alpha(x, y)| \leq M |x - y|^\gamma - |\alpha|.$$
holds for any \( x, y \in F \). Since for \( f \in \text{Lip} (\gamma, F) \), the collection \( \{ f^{(\alpha)} \} \) may not be unique, we could view \( \{ f^{(\alpha)} \}_{|\alpha| \leq k} \) as an element in \( \text{Lip} (\gamma, F) \) and take the smallest \( M \) as its norm.

Let \( \{ f^{(\alpha)} \}_{|\alpha| \leq k} \in \text{Lip} (\gamma, F) \). We define

\[
E_k \left( \{ f^{(\alpha)} \} \right) = \begin{cases} f^{(0)}(x) & \text{if } x \in F, \\ \sum_{j=1}^{\infty} P(x, p_j) \varphi_j^*(x) & \text{if } x \in F^c \end{cases}
\]

where

\[
P(x, y) = \sum_{|\alpha| \leq k} \frac{f^{\alpha}(y)}{\alpha!} (x - y)^\alpha, \quad x \in F^c, \quad y \in F.
\]

**Theorem 58.** The linear extension operator \( E_k \) maps \( \text{Lip} (\gamma, F) \) continuously into \( \text{Lip} (\gamma, \mathbb{R}^n) \) if \( k < \gamma \leq k + 1 \). The norm of this mapping has a bound independent of the closed set \( F \).

### 8.3. Extension theorem for Sobolev functions in Lipschitz domain.

Let \( k \geq 0 \) be an integer and \( 1 \leq p \leq \infty \). Similar to \( W^{k, p}(\mathbb{R}^n) \), we can define \( W^{k, p}(\Omega) \) for any open subset \( \Omega \subset \mathbb{R}^n \). The goal of this section is to find an extension operator from \( W^{k, p}(\Omega) \) to \( W^{k, p}(\mathbb{R}^n) \).

We first consider special Lipschitz domains whose boundary are represented by uniform Lipschitz graphs.

Let \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function satisfying

\[
|\varphi(x) - \varphi(x')| \leq M |x - x'| \quad \text{for any } x, x' \in \mathbb{R}^n.
\]

Let

\[
\Omega = \{ (x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, y \in \mathbb{R}, y > \varphi(x) \}.
\]

The smallest such \( M \) is called the bound of the special Lipschitz domain \( \Omega \).

**Lemma 22.** There exists a continuous function \( \psi \) defined on \([1, \infty)\) which is rapidly decreasing at \( \infty \) and satisfies

\[
\int_1^\infty \psi(t) \, dt = 1, \\
\int_1^\infty t^k \psi(t) \, dt = 0 \quad \text{for } k \in \mathbb{N}.
\]

**Proof.** Let

\[
\psi(t) = \frac{e}{\pi t} \text{Im} \left( e^{-\omega(t-1)^{1/2}} \right)
\]

where \( \omega = e^{-\frac{\pi}{4}} \). \( \square \)

**Lemma 23.** Let \( F = D \) and \( \Delta(x, y) \) be the regularized distance from \( F \). Then for any \( (x, y) \in F^c \),

\[
\varphi(x) - y \leq c \Delta(x, y)
\]

where \( c \) is a constant depending only on \( M \).

**Proof.** For any \( (x, y) \in F^c \),

\[
\delta(x, y) \geq \left( 1 + \frac{1}{M^2} \right)^{-\frac{1}{2}} (\varphi(x) - y).
\]
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Since
\[ \Delta (x, y) \geq \frac{1}{5} \delta (x, y), \]
the lemma holds with
\[ c = 5 \left( 1 + \frac{1}{M^2} \right)^{\frac{1}{2}} \]
\[ \square \]

Let
\[ \delta^* (x, y) = 2c \Delta (x, y). \]

We define for any \( f \in W^{k,p} (\Omega) \cap C^\infty (\bar{\Omega}) \) such that \( f \) and all its partial derivatives are bounded in \( \Omega \),
\[ E (f) (x, y) = \begin{cases} f (x, y) & \text{if } y \geq \varphi (x), \\ \int_1^\infty \psi (t) f (x, y + t \delta^* (x, y)) \, dt & \text{if } y < \varphi (x). \end{cases} \]

**Proposition 22.** \( E (f) \in C^\infty (\mathbb{R}^{n+1}) \) and
\[ \| E (f) \|_{W^{k,p} (\mathbb{R}^{n+1})} \leq A_{k,n} (M) \| f \|_{W^{k,p} (\Omega)}. \]

**Proof.** First, \[ \square \]

**Theorem 59.** The extension operator \( E \) maps \( W^{k,p} (\Omega) \) continuously into \( W^{k,p} (\mathbb{R}^{n+1}) \).

**Proof.** Let \( \eta \) be a nonnegative smooth function with compact support in the interior of the cone
\[ \Gamma_+ = \{ (x, y) : M |x| < |y|, y < 0 \}. \]
s.t.
\[ \int_{\mathbb{R}^{n+1}} \eta = 1. \]
For any \( \varepsilon > 0 \), define
\[ \eta_\varepsilon (u) = \frac{1}{\varepsilon^{n+1}} \eta \left( \frac{u}{\varepsilon} \right), u \in \mathbb{R}^{n+1}. \]

For any \( f \in W^{k,p} (\Omega) \), we define
\[ f_\varepsilon (u) = f \ast \eta_\varepsilon = \int_{\mathbb{R}^{n+1}} f (u - v) \eta_\varepsilon (v) \, dv. \]

Then \( f_\varepsilon \in C^\infty (\bar{\Omega}) \) and
\[ \| f_\varepsilon \|_{W^{k,p} (\Omega)} \leq \| f \|_{W^{k,p} (\Omega)}. \]

Now if \( 1 \leq p < \infty \), then
\[ \lim_{\varepsilon \to 0^+} \| f_\varepsilon - f \|_{W^{k,p} (\Omega)} = 0. \]

Let
\[ E_\varepsilon (f) = E (f_\varepsilon). \]

Then \( E_\varepsilon (f) \) is a Cauchy sequence in \( W^{k,p} (\mathbb{R}^{n+1}) \) and hence
\[ E (f) = \lim_{\varepsilon \to 0^+} E_\varepsilon (f) \]
defines an extension operator satisfying
\[ \| E (f) \|_{W^{k,p} (\mathbb{R}^{n+1})} \leq A_{k,n} (M) \| f \|_{W^{k,p} (\Omega)}. \]
And if $p = \infty$, $k \geq 1$, 
\[ \lim_{\varepsilon \to 0^+} \| f_\varepsilon - f \|_{W^{k-1,\infty}(\Omega)} = 0. \]
So $E_\varepsilon(f)$ is a Cauchy sequence in $W^{k-1,\infty}(\mathbb{R}^{n+1})$ and hence 
\[ E(f) = \lim_{\varepsilon \to 0^+} E_\varepsilon(f) \]
defines an extension operator satisfying 
\[ \| E(f) \|_{W^{k,\infty}(\mathbb{R}^{n+1})} \leq A_{k,n}(M) \| f \|_{W^{k,\infty}(\Omega)}. \]