1. Let \( f \in L^p(S^1) = L^p(-\pi, \pi), 1 < p < \infty \). Define the Hilbert transform

\[
Sf(y) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \frac{\cot (x/2)}{2\pi} f(x - y) \, dx.
\]

Show that

\[
\|Sf\|_{L^p(S^1)} \leq A_p \|f\|_{L^p(S^1)}
\]

for some constant \( A_p \) independent of \( f \). (Hint: Using the result of last homework.)

Proof:

\[
Sf(y) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \frac{\cot (x/2)}{2\pi} f(x - y) \, dx
\]

\[
= \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \frac{1}{\pi x} f(x - y) \, dx + \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \left( \frac{\cot (x/2)}{2\pi} - \frac{1}{\pi x} \right) f(x - y) \, dx
\]

\[
= Af + Bf.
\]

From last homework,

\[
\|Af\|_{L^p(S^1)} \leq A_p \|f\|_{L^p(S^1)}.
\]

Now,

\[
Bf = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq \pi} \left( \frac{\cot (x/2)}{2\pi} - \frac{1}{\pi x} \right) f(x - y) \, dx
\]

\[
= \int_{0 < |x| \leq \pi} \left( \frac{\cot (x/2)}{2\pi} - \frac{1}{\pi x} \right) f(x - y) \, dx
\]

since

\[
\left| \frac{\cot (x/2)}{2\pi} - \frac{1}{\pi x} \right| \leq c \text{ for any } 0 < |x| \leq \pi.
\]

Hence

\[
\|Bf\|_{L^p(S^1)} \leq (2\pi)^{\frac{1}{2}} \|Bf\|_{L^\infty(S^1)}
\]

\[
\leq (2\pi)^{\frac{1}{2}} c \|f\|_{L^1(S^1)}
\]

\[
\leq (2\pi)^{\frac{1}{2}} (2\pi)^{1-\frac{1}{p}} c \|f\|_{L^p(S^1)}
\]

\[
= 2\pi c \|f\|_{L^p(S^1)}.
\]

So we have

\[
\|Sf\|_{L^p(S^1)} \leq \|Af\|_{L^p(S^1)} + \|Bf\|_{L^p(S^1)} \leq A_p \|f\|_{L^p(S^1)}.
\]
2. Let \( T = (T_1, T_2, \cdots, T_n) \) where \( T_j, j = 1, 2, \cdots, n, \) are bounded operators on \( L^2(\mathbb{R}^n) \) and suppose
\[
\hat{T}f = m\hat{f}
\]
holds for some \( m \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \). Show that the following two statements are equivalent:
(a) For every rotation \( \rho \) of \( \mathbb{R}^n \) about the origin,
\[
\rho (m(x)) = m(\rho x).
\]
(b) For \( j = 1, 2, \cdots, n, \)
\[
\rho T_j \rho^{-1} f = \sum_{k=1}^n \rho_{jk} T_k f
\]
holds for any rotation \( \rho \) of \( \mathbb{R}^n \) about the origin.
Answer: Assuming (a) holds, since
\[
\hat{\rho f} = \rho \hat{f},
\]
we have
\[
\rho \hat{T_j \rho^{-1} f}(y) = \hat{T_j \rho^{-1} f}(\rho y) = m_j(\rho y) \rho^{-1} f(\rho y)
\]
\[
= \sum_{k=1}^n \rho_{jk} m_k(y) \hat{f}(y) = \sum_{k=1}^n \rho_{jk} \hat{T_k f}(y).
\]
Hence
\[
\rho T_j \rho^{-1} f = \sum_{k=1}^n \rho_{jk} T_k f.
\]