Sixth Homework Assignment for Math 2304

Due Date: Friday, February 18, 2011

1. Let $E$ be a closed set in $\mathbb{R}^n$ such that $|E^c| < \infty$, $\lambda > 0$ and

$$I^\lambda(x) = \int_{\mathbb{R}^n} \frac{\delta^\lambda(x + y)}{|y|^{n+\lambda}} dy.$$

Show that

$$\int_E I^\lambda(x) \, dx \leq c|E^c|$$

where $c$ is a constant depending only on $n$ and $\lambda$.

Answer:

$$\int_E I^\lambda(x) \, dx = \int_E \int_{\mathbb{R}^n} \frac{\delta^\lambda(x + y)}{|y|^{n+\lambda}} dy \, dx$$

$$= \int_E \int_{E^c} \frac{\delta^\lambda(y)}{|x - y|^{n+\lambda}} dy \, dx$$

$$= \int_{E^c} \delta^\lambda(y) \left( \int_E \frac{1}{|x - y|^{n+\lambda}} dx \right) dy$$

$$\leq \int_{E^c} \delta^\lambda(y) \left( \int_{|x| \geq \delta(y)} \frac{1}{|x|^{n+\lambda}} dx \right) dy$$

$$= \frac{n\omega_n}{\lambda} \int_{E^c} dy = \frac{n\omega_n}{\lambda} |E^c|.$$

2. (1) Let $\varphi_\varepsilon$ be the standard mollifier. Show that for any finite Borel signed measure $\mu \in M(\mathbb{R}^n)$,

$$\lim_{\varepsilon \to 0} \|\varphi_\varepsilon * \mu\| = \|\mu\|.$$

(2) Let $T : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ be defined by $Tf = f * \mu$. Show that

$$\|T\| = \|\mu\|.$$

Answer: (1) For any $f \in C_0(\mathbb{R}^n)$,

$$(\varphi_\varepsilon * \mu)(f) = \int_{\mathbb{R}^n} f(x + y) \varphi_\varepsilon(x) \, d\mu(y)$$

$$\leq \|\mu\| \|f\|_\infty \int_{\mathbb{R}^n} \varphi_\varepsilon(x) \, dx = \|\mu\| \|f\|_\infty.$$

Hence,

$$\|\varphi_\varepsilon * \mu\| \leq \|\mu\|.$$
On the other hand, Riesz representation implies that for any $\delta > 0$, there exists $f \in C_0(\mathbb{R}^n)$ s.t. $\|f\|_\infty \leq 1$ and

$$\|\mu\| \leq \int_{\mathbb{R}^n} f(x) \, d\mu(x) + \delta.$$  

Since $\varphi \ast f \to f$ uniformly as $\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ s.t. for any $\varepsilon < \varepsilon_0$, $\|\varphi \ast \tilde{f} - \tilde{f}\| \leq \delta$. Hence for any $\varepsilon < \varepsilon_0$,

$$\|\varphi \ast \mu\| \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x+y) \varphi \ast f(x) dxd\mu(y) = \int_{\mathbb{R}^n} \left( \varphi \ast \tilde{f} \right)(-y) d\mu(y)$$
$$= \int_{\mathbb{R}^n} f(x) d\mu(x) + \int_{\mathbb{R}^n} \left[ \left( \varphi \ast \tilde{f} \right)(-y) - f(y) \right] d\mu(y)$$
$$\geq \|\mu\| - \delta - \delta \|\mu\|.$$  

So we have

$$\lim_{\varepsilon \to 0} \|\varphi \ast \mu\| = \|\mu\|.$$  

(2) For any $\varphi \in C_0(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$, we have

$$\left| \int_{\mathbb{R}^n} Tf(x) \varphi(x) \, dx \right|$$
$$\leq \|\varphi\|_\infty \|f\|_1 \|\mu\|,$$

hence,

$$\|Tf\|_1 \leq \|f\|_1 \|\mu\|$$
and

$$\|T\| \leq \|\mu\|.$$  

Now since

$$\lim_{\varepsilon \to 0} \|T \varphi \ast \mu\|_1 = \|\mu\| = \|\mu\| \|\varphi \ast \mu\|_1,$$

we must have

$$\|T\| = \|\mu\|.$$  