Final for Math 2301   Fall 2009

The due date for this assignment is Friday, December 18.

1. Show that for any $r > 0$,

$$\int_0^\infty \left(\int_0^r e^{-xy^2} \sin x dx \right) dy = \int_0^r \left(\int_0^\infty e^{-xy^2} \sin x dy \right) dx.$$

   Use this identity to prove

$$\int_0^\infty \sin\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

2. Let $1 \leq p \leq \infty$. If g is Lebesgue measurable on $\mathbb{R}$ and $fg \in L^1(\mathbb{R})$ for any $f \in L^p(\mathbb{R})$, then $g \in L^q(\mathbb{R})$ where q is the Hölder conjugate of p.

3. Assume $f \in L^1(\mathbb{R})$, show that

$$\lim_{h\to 0} \int_a^b |f(x+h) - f(x)| \, dx = 0.$$

4. Let $\alpha \in \mathbb{R}$. Find all values of α so that

$$f_\alpha(x) = \begin{cases} x^\alpha \sin \frac{1}{x} & \text{if} \quad 0 < x \leq 1, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

   is a function of bounded variation on $[0,1]$.

5. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuous curve. The length of γ is defined by

$$L(\gamma) = \sup \left\{ \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| : P = \{t_0, t_1, \cdots, t_n\} \text{ is a partition of } [a,b] \right\}.$$

   Writing $\gamma = (\gamma_1, \cdots, \gamma_n)$, show that $L(\gamma) < \infty$ if and only if $\gamma_k \in BV[a,b]$ for any $k = 1, \cdots, n$.