Assignment 9 for Math 2301 Fall 2009

The due date for this assignment is Friday, November 6.

1. (a). Let (X,μ) be a measure space with $\mu(X)<\infty$. Let $1\le p<q\le\infty$, show that
$$L^q(\mu)\subset L^p(\mu).$$
(Hint: Using Holder's inequality)
(b). Construct a function $f\in L^2(\mathbb{R})$ such that $f\notin L^p(\mathbb{R})$ for any $p\neq 2$.
Proof: (a). For any $f\in L^q(\mu)$, we have from Hölder's inequality
$$\int_X |f|^p\,d\mu=\int_X |f|^p\cdot 1d\mu\le\left(\int_X |f|^q\,d\mu\right)^{\frac{p}{q}}\left(\int_X 1d\mu\right)^{1-\frac{p}{q}}=\|f\|_q^p\left(\mu(X)\right)^{1-\frac{p}{q}}<\infty,$$
where the Hölder conjugates we used are
$$\alpha=\frac{q}{p},\beta=\frac{1}{1-\frac{p}{q}}=\frac{q}{q-p}.$$
Hence $f\in L^p(\mu)$. Here a good property we have proved is that
$$(\mu(X))^{-\frac{1}{p}}\|f\|_p=\left(\frac{1}{\mu(X)}\int_X |f|^p\,d\mu\right)^{\frac{1}{p}}$$
is monotone increasing in p.
(b). We define
$$f(x)=\begin{cases}\frac{1}{|x|^{\frac{1}{2}}\ln(|x|)} & \text{if} \quad 0<|x|<\frac{1}{e}\text{ or } |x|>e,\\ 0 & \text{otherwise.}\end{cases}$$
Then $f\in L^2(\mathbb{R})$ and $f\notin L^p(\mathbb{R})$ for any $p\neq 2$.

2. Let (X,μ) be a measure space and $1\le p<r<q\le\infty$. If $f\in L^p(\mu)\cap L^q(\mu)$, show that $f\in L^r(\mu)$.
Proof: We first assume $q<\infty$. For any $f\in L^r(\mu)$. Let $\alpha,\beta>0$ be such that
$$\alpha+\beta=r,\frac{\alpha}{p}+\frac{\beta}{q}=1,$$
i.e.,
$$\alpha=\frac{p(q-r)}{q-p},\beta=\frac{q(r-p)}{q-p}.$$
Then we have from Hölder's inequality
$$\int_X |f|^r\,d\mu=\int_X |f|^\alpha\cdot|f|^\beta\,d\mu\le\left(\int_X |f|^p\,d\mu\right)^{\frac{\alpha}{p}}\left(\int_X |f|^q\,d\mu\right)^{\frac{\beta}{q}}<\infty,$$
hence $f\in L^r(\mu)$. If $q=\infty$, we have
$$\int_X |f|^r\,d\mu=\int_X |f|^p\cdot|f|^{r-p}\,d\mu\le\|f\|_\infty^{r-p}\int_X |f|^p\,d\mu<\infty,$$
so we still have $f\in L^r(\mu)$.

3. Let (X, μ) be a measure space with $\mu(X) < \infty$ and $f \in L^{\infty}(\mu)$. Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty .$$

Proof: First, we have $|f| \leq \|f\|_\infty$ μ-a.e.. Hence,

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \leq \|f\|_\infty \, \mu(X)^{\frac{1}{p}} .$$

And we have

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty . \tag{1}$$

On the other hand, for any $\varepsilon \in (0, \|f\|_\infty)$, we have

$$E_\varepsilon = \{x \in X : |f|(x) \geq \|f\|_\infty - \varepsilon\}$$

is measurable and $\mu(E_\varepsilon) > 0$. Hence

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \geq \left(\int_{E_\varepsilon} |f|^p \, d\mu\right)^{\frac{1}{p}} \geq (\|f\|_\infty - \varepsilon)\, \mu(E_\varepsilon)^{\frac{1}{p}} ,$$

and let $p \to \infty$, we have

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty - \varepsilon.$$

Since ε can be arbitrarily small, we have

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty . \tag{2}$$

(1) and (2) imply

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty .$$