

Assignment 12 for Math 2301 Fall 2009

The due date for this assignment is Friday, December 4.

1. Let $E \subset [0, 1]$ be the Cantor set created by repeatedly deleting the open middle thirds of a set of line segments. Let $I_1^n, \dots, I_{2^n-1}^n$ be the 2^n-1 open intervals removed in the n -th step with the right order. Now define a function $f : [0, 1] \rightarrow [0, 1]$ such that
 - (i). $f(0) = 0$;
 - (ii). if $x \in I_i^n$ for some $1 \leq i \leq 2^n-1$, then $f(x) = \frac{2i-1}{2^n}$;
 - (iii). if $x \in C$ and $x \neq 0$, then

$$f(x) = \sup \{f(t) : t < x \text{ and } t \in [0, 1] \setminus C\}.$$

Show that f is continuous, monotone increasing and $f' = 0$ a.e..

Proof: Step 1: We claim that if we list the $2^n - 1$ intervals removed in the first n steps in the natural order: $J_j^n, 1 \leq j \leq 2^n - 1$, then for any $t \in J_j^n, f(t) = \frac{j}{2^n}$. We prove this by induction, when $n = 1, J_1^1 = I_1^1$, then for any $t \in J_1^1, f(t) = \frac{2 \cdot 1 - 1}{2} = \frac{1}{2}$. Assume it is true when $n = k$, we have $J_j^k, 1 \leq j \leq 2^k - 1$, then when $n = k + 1$, we removed another 2^k intervals, $I_i^{k+1}, 1 \leq i \leq 2^k$, if we list the $2^{k+1} - 1$ intervals as

$$J_j^{k+1}, 1 \leq j \leq 2^{k+1} - 1,$$

then from the construction of Cantor set, we have, $I_i^{k+1} = J_{2i-1}^{k+1}$ and $J_j^k = J_{2j}^{k+1}$, hence, we have for any $t \in J_j^{k+1}$, if $j = 2i - 1$ is odd, then $t \in I_i^{k+1}, f(t) = \frac{2i-1}{2^{k+1}} = \frac{j}{2^{k+1}}$. If $j = 2i$ is even, then $t \in J_i^k$, hence $f(t) = \frac{i}{2^k} = \frac{j}{2^{k+1}}$. Hence, the claim is true when $n = k + 1$.

Step 2: f is monotone increasing. For any $0 \leq x_1 < x_2 \leq 1$,

Case I: $x_1 = 0$, then $f(x_2) \geq 0 = f(x_1)$.

Case II: $x_1, x_2 \in [0, 1] \setminus C$, then there exists N such that $x_1 \in J_i^N, x_2 \in J_j^N$ with $i \leq j$, hence $f(x_1) = \frac{i}{2^N} \leq \frac{j}{2^N} = f(x_2)$.

Case III: $x_1, x_2 \in C$ and $x_1 \neq 0$, then since

$$\{f(t) : t < x_1 \text{ and } t \in [0, 1] \setminus C\} \subset \{f(t) : t < x_2 \text{ and } t \in [0, 1] \setminus C\},$$

we have $f(x_1) \leq f(x_2)$ from (iii).

Case IV: $x_1 \in [0, 1] \setminus C$ and $x_2 \in C$, then since $x_1 < x_2$,

$$f(x_1) \leq \sup \{f(t) : t < x_2 \text{ and } t \in [0, 1] \setminus C\} = f(x_2).$$

Case V: $x_1 \in C, x_1 \neq 0$ and $x_2 \in [0, 1] \setminus C$, then for any $t < x_1$ and $t \in [0, 1] \setminus C$, since $t < x_2$, Case II implies $f(t) \leq f(x_2)$, hence,

$$f(x_1) = \sup \{f(t) : t < x_1 \text{ and } t \in [0, 1] \setminus C\} \leq f(x_2).$$

Step 3: Since f is constant on open intervals $I_k^n, f' = 0$ on

$$[0, 1] \setminus C = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{2^n-1} I_k^n \right).$$

Since C has Lebesgue measure zero, we have $f' = 0$ a.e. on $[0, 1]$.

Step 4: f is continuous. If $x \in [0, 1] \setminus C$, then f is a constant in a neighborhood of x , and hence f is continuous at x . If $x \in C$, for each $n \in \mathbb{N}$, let $J_j^n = (a_j, b_j)$, $1 \leq j \leq 2^n - 1$ intervals removed in the first n steps listed in the natural order, then we have $|J_j^n| \geq \frac{1}{3^n}$. Let $\delta = \frac{1}{3^n}$. For any $|x' - x| < \frac{1}{3^n}$, $x \in [0, 1]$,

Case I: x is between $J_j^n = (a_j, b_j)$ and $J_{j+1}^n = (a_{j+1}, b_{j+1})$, then $x, x' \in (a_j, b_{j+1})$, so we have

$$\frac{j}{2^n} \leq f(x), f(x') \leq \frac{j+1}{2^n};$$

Case II: $0 \leq x \leq a_1$, then $x, x' \in [0, b_1)$, so we have

$$0 \leq f(x), f(x') \leq \frac{1}{2^n};$$

Case III: $b_{2^n-1} \leq x \leq 1$, then $x, x' \in (a_{2^n-1}, 1]$, so we have

$$\frac{2^n - 1}{2^n} \leq f(x), f(x') \leq 1.$$

Hence, in any case, we have

$$|f(x) - f(x')| \leq \frac{1}{2^n}.$$

Thus f is continuous on $[0, 1]$.

2. Let $f, g \in BV[a, b]$ and $g \geq 1$ on $[a, b]$. Show that $\frac{f}{g} \in BV[a, b]$.

Proof: Since $\frac{1}{x}$ is Lipschitz continuous on $[1, \infty)$ and $g \in BV[a, b]$, we have $\frac{1}{g} \in BV[a, b]$.

Since $f \in BV[a, b]$, we have $\frac{f}{g} = f \cdot \frac{1}{g} \in BV[a, b]$.