Assignment 1 for Math 2301 Fall 2009

The due date for this assignment is Wednesday, September 9.

1. **Let X be a nonempty set, and let $\mathcal{F}$ be a collection of subsets in X. Show that any element of the σ-algebra generated by $\mathcal{F}$ belongs to the σ-algebra generated by some countable subcollection of $\mathcal{F}$.
   Proof: It suffices to show
   $$\sigma(\mathcal{F}) = \bigcup_\alpha \sigma(\mathcal{F}_\alpha)$$
   where each $\mathcal{F}_\alpha$ is a countable subcollection of $\mathcal{F}$ and the union is taken on all such $\mathcal{F}_\alpha$. Since $\mathcal{F}_\alpha \subset \mathcal{F}$, we have $\sigma(\mathcal{F}) \supset \sigma(\mathcal{F}_\alpha)$ for each α, and hence
   $$\sigma(\mathcal{F}) \supset \bigcup_\alpha \sigma(\mathcal{F}_\alpha).$$
   Next, we show $\bigcup_\alpha \sigma(\mathcal{F}_\alpha)$ is a σ-algebra. It is easy to see that $X \in \bigcup_\alpha \sigma(\mathcal{F}_\alpha)$, and $\bigcup_\alpha \sigma(\mathcal{F}_\alpha)$ is closed under complements. Now let $A_k \in \bigcup_\alpha \sigma(\mathcal{F}_\alpha)$, $k = 1, 2, \cdots$. For each k, there exists a countable subcollection of $\mathcal{F}$, which we denoted by $\mathcal{F}_k$, such that $A_k \in \sigma(\mathcal{F}_k)$. Let $\mathcal{F}_0 = \bigcup_{k=1}^{\infty} \mathcal{F}_k$, then $\mathcal{F}_0$ is a countable subcollection of $\mathcal{F}$, so we have
   $$A = \bigcup_{k=1}^{\infty} A_k \in \sigma(\mathcal{F}_0) \subset \bigcup_\alpha \sigma(\mathcal{F}_\alpha).$$
   Hence, $\bigcup_\alpha \sigma(\mathcal{F}_\alpha)$ is a σ-algebra containing $\mathcal{F}$, and we have
   $$\sigma(\mathcal{F}) \subset \bigcup_\alpha \sigma(\mathcal{F}_\alpha).$$
   This finishes the proof.

2. Let X be an uncountable set, and $\mathcal{F}$ is the collection of one-point subsets of X. Prove that the σ-algebra generated by $\mathcal{F}$ is
   $$\mathcal{M} = \{E \subset X : E \text{ or } E^c \text{ is at most countable.}\}$$
   Proof: It suffices to verify two thing: 1. $\mathcal{M} \subset \sigma(\mathcal{F})$; 2. $\mathcal{M}$ is a σ-algebra.

3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers. Define $\mu : 2^{\mathbb{N}} \to [0, \infty]$ by $\mu(\emptyset) = 0$ and for each nonempty set A of $\mathbb{N}$,
   $$\mu(A) = \sum_{n \in A} a_n.$$

Show that μ is a measure.

Proof: We only need to check countable additivity. Let $A_k \subset \mathbb{N}$, $k = 1, 2, \cdots$ be pairwise disjoint, and $A = \bigcup_{k=1}^{\infty} A_k$, we need to prove

$$\mu(A) = \sum_{k=1}^{\infty} \mu(A_k),$$

i.e.,

$$\sum_{n \in A} a_n = \sum_{k=1}^{\infty} \sum_{n \in A_k} a_n. \tag{1}$$

Since a_n is nonnegative, rearrangements of an infinite series will not change its sum and (1) follows.

4. **Let μ^* be an outer measure on X. Assume that a subset A of X has the property that for each $\varepsilon > 0$, there exists a measurable set E such that $\mu^*(A \triangle E) < \varepsilon$. Show that A is also measurable. Here $A \triangle E$ is symmetric difference.

   Proof: For any subset B of X, we need to prove

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \backslash A). \tag{2}$$

   For each $\varepsilon > 0$, there exists a measurable set E such that $\mu^*(A \triangle E) < \varepsilon$. For such E, we have

$$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \backslash E).$$

   Since $B \cap A \subset (B \cap E) \cup (A \triangle E)$, and $B \backslash A \subset (B \backslash E) \cup (A \triangle E)$, we have

$$\mu^*(B \cap A) \leq \mu^*(B \cap E) + \mu^*(A \triangle E) < \mu^*(B \cap E) + \varepsilon,$$
$$\mu^*(B \backslash A) \leq \mu^*(B \backslash E) + \mu^*(A \triangle E) < \mu^*(B \backslash E) + \varepsilon.$$

   Hence,

$$\begin{aligned} \mu^*(B) &\geq \mu^*(B \cap E) + \mu^*(B \backslash E) \\ &\geq \mu^*(B \cap A) + \mu^*(B \backslash A) - 2\varepsilon. \end{aligned}$$

   (2) follows by letting $\varepsilon \to 0$.

5. Let μ^* be an outer measure on X. Show that for every measurable set E and every subset $A \subset X$, we have

$$\mu^*(E) + \mu^*(A) = \mu^*(E \cup A) + \mu^*(E \cap A). \tag{3}$$

   Proof: Since E is measurable, we have

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E) + \mu^*(A \backslash E), \\ \mu^*(E \cup A) &= \mu^*((E \cup A) \cap E) + \mu^*((E \cup A) \backslash E) \\ &= \mu^*(E) + \mu^*(A \backslash E). \end{aligned}$$

Hence,

$$\mu^*(E) + \mu^*(A) + \mu^*(A \backslash E) = \mu^*(E \cup A) + \mu^*(E \cap A) + \mu^*(A \backslash E).$$

(3) follows if $\mu^*(A \backslash E) < \infty$. If $\mu^*(A \backslash E) = \infty$, we have $\mu^*(A) = \mu^*(E \cup A) = \infty$, so (3) still holds.