Answer to some of the Sample Problems for Second Midterm for Math 0230
November 5, 2008

1. Determine the radius of convergence and interval of convergence for

(a)

$$\sum_{n=0}^{\infty} \frac{(2x-3)^{3n}}{3^{2n+1}};$$

Answer: We use ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-3|^3}{3^2}.$$

So when

$$\frac{|2x-3|^3}{3^2} < 1,$$

i.e.,

$$\left|x - \frac{3}{2}\right| < \frac{3^{\frac{2}{3}}}{2},$$

the series converges. And when

$$\left|x - \frac{3}{2}\right| > \frac{3^{\frac{2}{3}}}{2},$$

the series diverges. So the radius of convergence is $R = \frac{3^{\frac{2}{3}}}{2}$. It is straight forward to check that at both end points, the series diverges by divergence test, so the interval of convergence is

$$\left|x - \frac{3}{2}\right| < \frac{3^{\frac{2}{3}}}{2},$$

i.e., $\left(\frac{3}{2} - \frac{3^{\frac{2}{3}}}{2}, \frac{3}{2} + \frac{3^{\frac{2}{3}}}{2}\right)$.

(b)

$$\sum_{n=0}^{\infty} \frac{x^n}{2n+1};$$

Answer: We use ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(2n+1)|x|}{2(n+1)+1} = |x|.$$

So when

$$|x| < 1,$$

the series converges. And when

$$|x| > 1,$$

the series diverges. So the radius of convergence is $R = 1$. Now we check the end points. When $x = 1$, we have

$$\sum_{n=0}^{\infty} \frac{1}{2n+1}$$

which diverges by comparing the harmonic series; when $x = -1$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

which converges by the alternating series test. So the interval of convergence is $[-1, 1)$.

(c)

$$\sum_{n=0}^{\infty} (-1)^n n^2 2^n (x+1)^n .$$

Answer: We use ratio test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2(n+1)^2 |x+1|}{n^2} = 2|x+1| .$$

So when

$$2|x+1| < 1,$$

i.e.,

$$|x+1| < \frac{1}{2},$$

the series converges. And when

$$|x+1| > \frac{1}{2},$$

the series diverges. So the radius of convergence is $R = \frac{1}{2}$. Now we check the end points, when $x = -2$ and when $x = 0$, both series diverge by divergence test, so the interval of convergence is $(-2, 0)$.

2. Determine the Taylor series about $x = 0$ and find its radius convergence for

(a)

$$\frac{1}{2x+3};$$

Answer:

$$\frac{1}{2x+3} = \frac{1}{3}\frac{1}{1-(-\frac{2x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{3^{n+1}}x^n.$$

Here $R = \frac{3}{2}$.

(b)

$$\sqrt{1-2x};$$

Answer:

$$\sqrt{1-2x} = (1+(-2x))^{\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-2x)^n = \sum_{n=0}^{\infty}(-2)^n\binom{\frac{1}{2}}{n}x^n.$$

Now when $n \geq 2$,

$$(-2)^n\binom{\frac{1}{2}}{n} = (-2)^n\frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)\times\cdots\left(\frac{1}{2}-n+1\right)}{n!}$$
$$= \frac{-1\times 3\times\cdots(2n-3)}{n!}$$

so we have

$$\sqrt{1-2x} = 1-2x-\frac{1}{2}x^2-\frac{1\times 3}{3!}x^3-\frac{1\times 3\times 5}{5!}x^5-\cdots-\frac{1\times 3\times\cdots(2n-3)}{n!}x^n-\cdots$$
$$= 1-2x-\sum_{n=2}^{\infty}\frac{1\times 3\times\cdots(2n-3)}{n!}x^n.$$

Here $R = \frac{1}{2}$.

(c)

$$x^2\tan^{-1}x.$$

Answer:

$$x^2\tan^{-1}x = x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+3}.$$

Here $R = 1$.

3. Let $f(x) = e^x \sin x^7$, find $f^{(10)}(0)$.
Answer:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots,$$
$$\sin x^7 = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^7\right)^{2n+1}}{(2n+1)!} = x^7 - \frac{1}{6}x^{21} + \cdots,$$

hence,

$$e^x \sin x^7 = x^7 + x^8 + \frac{x^9}{2} + \frac{x^{10}}{6} + \cdots.$$

So the coefficient of x^{10} is

$$\frac{1}{6} = \frac{f^{(10)}(0)}{10!},$$

and we get

$$f^{(10)}(0) = \frac{10!}{6}.$$