Answer to First Midterm for Math 230
October 1, 2008

1. Evaluate the integral

(a) (10 points)

$$\int_0^\infty 2xe^{-3x}dx;$$

Solution: This is a type I improper integral.

$$\begin{aligned}
&\int_0^\infty 2xe^{-3x}dx \\
&= \lim_{t\to\infty}\int_0^t 2xe^{-3x}dx \\
&= \lim_{t\to\infty}\left(-\frac{2}{3}xe^{-3x}\Big|_0^t + \int_0^t \frac{2}{3}e^{-3x}dx\right) \\
&= \lim_{t\to\infty}\left(-\frac{2}{3}te^{-3t} - \frac{2}{9}e^{-3t} + \frac{2}{9}\right) = \frac{2}{9}.
\end{aligned}$$

(b) (10 points)

$$\int \frac{4x}{x^2-2x-3}dx;$$

Solution: We write the integrand into partial fractions.

$$\frac{4x}{x^2-2x-3} = \frac{4x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1},$$

hence,

$$4x = A(x+1) + B(x-3).$$

With $x = -1$, we obtain $B = 1$ and with $x = 3$, we obtain $A = 3$. So

$$\begin{aligned}
\int \frac{4x}{x^2-2x-3}dx &= \int \frac{3}{x-3} + \frac{1}{x+1}dx \\
&= 3\ln|x-3| + \ln|x+1| + C.
\end{aligned}$$

(c) (10 points)

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx.$$

Solution: Let $u = \sin x$, $du = \cos x dx$. We have

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx = \int_0^1 u^6\left(1-u^2\right)du = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}.$$

2. (10 points) Using Euler's method with step size $h = \frac{1}{3}$ to approximate $y(1)$ if

$$y' = 3x - y, y(0) = 3.$$

Solution: We have $x_0 = 0$, $x_1 = \frac{1}{3}$,$x_2 = \frac{2}{3}$, $x_3 = 1$. Now

$$\begin{aligned} y_0 &= 3, \\ y_1 &= y_0 + f(x_0, y_0) h = 3 + (0 - 3) \frac{1}{3} = 2, \\ y_2 &= y_1 + f(x_1, y_1) h = 2 + (1 - 2) \frac{1}{3} = \frac{5}{3}, \\ y_3 &= y_2 + f(x_2, y_2) h = \frac{5}{3} + \left(2 - \frac{5}{3}\right) \frac{1}{3} = \frac{16}{9}. \end{aligned}$$

3. (15 points) Find the length of the curve described by the parametric equation

$$x = 2 - 3t^2, y = 3 + 2t^3, 0 \le t \le 1.$$

Solution:

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \sqrt{(-6t)^2 + (6t^2)^2} dt \\ &= 6t\sqrt{1 + t^2} dt. \end{aligned}$$

Hence

$$\begin{aligned} L &= \int_0^1 6t\sqrt{1 + t^2} dt \\ &= 3 \int_1^2 \sqrt{u} du \\ &= 2u^{\frac{3}{2}} \Big|_1^2 = 2\left(2\sqrt{2} - 1\right). \end{aligned}$$

4. (15 points) Find the volume of the solid formed by rotating the region in the first quadrant bounded by $y = x^2$, $y = 2 - x$, and $x = 0$ about y-axis.
Solution: We use cylindrical shell method,

$$\begin{aligned} V &= \int 2\pi rl \\ &= \int_0^1 2\pi x \left(2 - x - x^2\right) dx \\ &= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{5\pi}{6}. \end{aligned}$$

5. (15 points) Solve the initial value problem

$$\frac{dy}{dt} = 3\left(1+t^2\right)\left(1+y^2\right),\ y\left(0\right) = 1.$$

Solution:

$$\int \frac{dy}{1+y^2} = \int 3\left(1+t^2\right) dt,$$

so

$$\tan^{-1} y = 3t + t^3 + C.$$

Then $y\left(0\right) = 1$ implies $C = \frac{\pi}{4}$. Hence,

$$\tan^{-1} y = 3t + t^3 + \frac{\pi}{4},$$

i.e.,

$$y = \tan\left(3t + t^3 + \frac{\pi}{4}\right).$$

6. (15 points) A tank full of water has the shape of a circular cylinder, with radius 2m and height 10m. Find the work needed to pump all the water out of the top. (The density of the water is $\rho = 1000$ kg/m^3 and the gravitational acceleration $g = 10$ m/sec^2.)
Solution: We cut the cylinder into n horizontal pieces,

$$\begin{aligned} W &\approx \sum_{i=1}^{n} W_i d_i \\ &= \sum_{i=1}^{n} \rho V_i g d_i \\ &\approx \sum_{i=1}^{n} \rho \pi r^2 \triangle x g x. \end{aligned}$$

Hence,

$$\begin{aligned} W &= \int_0^{10} \rho g \pi r^2 x dx = \int_0^{10} 1000 \cdot 10 \cdot \pi \cdot 2^2 x dx \\ &= 2 \times 10^6 \pi\ J. \end{aligned}$$