

Second Midterm for Math 230

November 5, 2008

1. Determine the convergence or divergence of the following series, show your reasoning.

(a) (10 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1 + \ln n};$$

Answer: This is alternating series, since

$$b_n = \frac{1}{1 + \ln n}$$

is monotone decreasing and

$$\lim_{n \rightarrow \infty} \frac{1}{1 + \ln n} = 0.$$

By alternating series test, the series converges.

- (b) (10 points) Using integral test to show convergence or divergence of

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n};$$

Answer: Let

$$f(x) = \frac{1}{x \ln x},$$

then $f(x)$ is positive and monotone decreasing when $x > 1$. So we can use integral test and consider the indefinite integral

$$\begin{aligned} & \int_e^{\infty} \frac{1}{x \ln x} dx \\ &= \lim_{t \rightarrow \infty} \int_e^t \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_1^{\ln t} \frac{1}{u} du \\ &= \lim_{t \rightarrow \infty} \ln(\ln t) = \infty. \end{aligned}$$

So the indefinite integral diverges and hence

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges by integral test.

(c) (10 points)

$$\sum_{n=0}^{\infty} \frac{3^{n+1} (n^3 + 1)}{2^{2n+1}}.$$

Answer: We use ratio test. Let

$$a_n = \frac{3^{n+1} (n^3 + 1)}{2^{2n+1}},$$

we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{3 \left((n+1)^3 + 1 \right)}{2^2 (n^3 + 1)} \\ &= \lim_{n \rightarrow \infty} \frac{3 \left(\left(1 + \frac{1}{n}\right)^3 + \frac{1}{n^3} \right)}{2^2 \left(1 + \frac{1}{n^3}\right)} = \frac{3}{4}. \end{aligned}$$

Since $\frac{3}{4} < 1$, the series converges by ratio test.

2. (10 points) Find all numbers p so that

$$\sum_{n=2}^{\infty} \frac{n+1}{(n^2+n+1)^p}$$

converges. Show your reasoning.

Answer: We use limiting comparison. Let

$$a_n = \frac{n+1}{(n^2+n+1)^p}, b_n = \frac{n}{n^{2p}} = \frac{1}{n^{2p-1}},$$

then both a_n, b_n are positive, and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n+1}{(n^2+n+1)^p}}{\frac{n}{n^{2p}}} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{\left(1 + \frac{1}{n} + \frac{1}{n^2}\right)^p} = 1. \end{aligned}$$

Since

$$\sum \frac{1}{n^{2p-1}}$$

converges if and only if

$$2p - 1 > 1,$$

i.e.,

$$p > 1.$$

The series

$$\sum_{n=2}^{\infty} \frac{n+1}{(n^2+n+1)^p}$$

converges when $p > 1$ and diverges when $p \leq 1$.

3. (15 points) Determine the radius of convergence and interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)3^{n+1}}.$$

Show your reasoning.

Answer: Let

$$a_n = \frac{(x-1)^n}{(n+1)3^{n+1}},$$

then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)}{3(n+2)} |x-1| = \frac{|x-1|}{3}.$$

By the ratio test, the power series converges when

$$\frac{|x-1|}{3} < 1,$$

i.e.,

$$|x-1| < 3,$$

and diverges when

$$|x-1| > 3,$$

so the radius of convergence $R = 3$. Now we consider the end points. $|x-1| = 3$ gives $x = -2$ or $x = 4$. When $x = -2$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

which is alternating harmonic series and hence converges. When $x = 4$, we get

$$\sum_{n=0}^{\infty} \frac{1}{n+1}$$

which is harmonic series and hence diverges. So the interval of convergence is

$$[-2, 4).$$

4. (15 points) Find the Taylor series centered at $a = 1$ for

$$f(x) = \ln(x).$$

Find its radius of convergence.

Answer:

$$\begin{aligned} \ln x &= \ln(x-1+1) = \ln(1 - (-(x-1))) \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} (-(x-1))^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n. \end{aligned}$$

We use ratio test to find its radius of convergence,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} |x-1| = |x-1|,$$

so the Taylor series converges when $|x-1| < 1$ and diverges when $|x-1| > 1$. Hence its radius of convergence is $R = 1$.

5. (15 points) Find all solutions to the second order nonhomogeneous differential equation

$$y'' - 4y' + 4y = 12x.$$

Answer: We first seek homogeneous solutions,

$$r^2 - 4r + 4 = 0,$$

we get repeated root $r = 2$, hence

$$y_h = (C_1 + C_2x)e^{2x}.$$

We then seek special solution using method of undetermined coefficients.

Let

$$y_p = ax + b,$$

we have

$$y_p' = a, y_p'' = 0,$$

so

$$-4a + 4(ax + b) = 12x.$$

Comparing the coefficients, we get

$$4a = 12, -4a + 4b = 0,$$

hence, $a = 3$ and $b = 3$. So

$$y_p = 3x + 3,$$

and the general solution is

$$y = y_h + y_p = (C_1 + C_2x)e^{2x} + 3x + 3.$$

6. (15 points) Solve the initial value problem:

$$y' + \frac{4x}{1+x^2}y = 1, \quad y(0) = 2008.$$

Answer: The integrating factor

$$I(x) = e^{\int \frac{4x}{1+x^2} dx} = (1+x^2)^2.$$

Multiplying both sides by $(1 + x^2)^2$, we have

$$\frac{d}{dx} \left((1 + x^2)^2 y \right) = (1 + x^2)^2.$$

Integrating the identity, we get

$$\begin{aligned} (1 + x^2)^2 y &= \int (1 + x^2)^2 dx \\ &= \int 1 + 2x^2 + x^4 dx \\ &= x + \frac{2}{3}x^3 + \frac{1}{5}x^5 + C, \end{aligned}$$

hence the general solution is

$$y = \frac{x + \frac{2}{3}x^3 + \frac{1}{5}x^5 + C}{(1 + x^2)^2}.$$

Next, the initial value $y(0) = 2008$ implies $C = 2008$, and hence

$$y = \frac{x + \frac{2}{3}x^3 + \frac{1}{5}x^5 + 2008}{(1 + x^2)^2}.$$