

First Midterm for Math 230

October 3, 2007

Last Name: \_\_\_\_\_ First Name: \_\_\_\_\_

Discussion Session(Your TA's name): \_\_\_\_\_

1. Evaluate the integral

(a) (10 points)

$$\int_{\frac{1}{2}}^1 \frac{3}{\sqrt{1-x^2}} dx;$$

Solution: This is a type II improper integral.

$$\begin{aligned} & \int_{\frac{1}{2}}^1 \frac{3}{\sqrt{1-x^2}} dx \\ &= \lim_{t \rightarrow 1^-} \int_{\frac{1}{2}}^t \frac{3}{\sqrt{1-x^2}} dx \\ &= \lim_{t \rightarrow 1^-} 3 \sin^{-1} x \Big|_{\frac{1}{2}}^t \\ &= \lim_{t \rightarrow 1^-} 3 \left( \sin^{-1} t - \sin^{-1} \frac{1}{2} \right) \\ &= 3 \left( \frac{\pi}{2} - \frac{\pi}{6} \right) = \pi. \end{aligned}$$

(b) (10 points)

$$\int_0^{\infty} 2xe^{-3x} dx;$$

Solution: This is a type I improper integral.

$$\begin{aligned} & \int_0^{\infty} 2xe^{-3x} dx \\ &= \lim_{t \rightarrow \infty} \int_0^t 2xe^{-3x} dx \\ &= \lim_{t \rightarrow \infty} \left( -\frac{2}{3}xe^{-3x} \Big|_0^t + \frac{2}{3} \int_0^t e^{-3x} dx \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{2}{3}te^{-3t} + -\frac{2}{9}e^{-3x} \Big|_0^t \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{2}{3}te^{-3t} - \frac{2}{9}e^{-3t} + \frac{2}{9} \right) \\ &= \frac{2}{9}. \end{aligned}$$

Here  $\lim_{t \rightarrow \infty} te^{-3t} = 0$  follows from L'Hospital rule.

(c) (10 points)

$$\int \frac{x}{x^2 + 3x + 2} dx;$$

Solution: We write the integrand into partial fractions,

$$\frac{x}{x^2 + 3x + 2} = \frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

hence,

$$x = A(x+2) + B(x+1),$$

and we obtain  $A = -1, B = 2$ . So we have

$$\begin{aligned} & \int \frac{x}{x^2 + 3x + 2} dx \\ &= \int \frac{-1}{x+1} + \frac{2}{x+2} dx \\ &= -\ln|x+1| + 2\ln|x+2| + C \\ &= \ln \frac{(x+2)^2}{|x+1|} + C. \end{aligned}$$

2. (15 points) Find the length of the curve

$$y = 1 + 2x^{\frac{3}{2}}, 0 \leq x \leq 4.$$

Solution:

$$\begin{aligned} L &= \int ds = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^4 \sqrt{1 + (3x^{\frac{1}{2}})^2} dx = \int_0^4 \sqrt{1 + 9x} dx \\ &= \frac{2}{3 \times 9} (1 + 9x)^{\frac{3}{2}} \Big|_0^4 = \frac{2}{27} (37\sqrt{37} - 1). \end{aligned}$$

3. (15 points) Set up an integral representing the volume of the solid formed by rotating the region bounded by  $y = (x-1)^2$  and  $y = 2x+1$  about  $y$ -axis. You are not required to evaluate the integral.

Solution: First we solve the intersection, and we get

$$(x-1)^2 = 2x+1,$$

and  $x_1 = 0, x_2 = 4$ . Hence two intersections are  $(0, 1)$  and  $(4, 9)$ . It is convenient to use cylindrical method, and we have

$$\begin{aligned} V &= \int 2\pi r l \\ &= \int_0^4 2\pi x (2x+1 - (x-1)^2) dx. \end{aligned}$$

4. (15 points) Solve the initial value differential equation

$$y' = x^2(y + 1), y(0) = 5.$$

Solution: This is separable, if  $y \neq -1$ , we have

$$\int \frac{dy}{y+1} = \int x^2 dx,$$

hence

$$\ln|y+1| = \frac{x^3}{3} + C,$$

and

$$y+1 = \pm e^{\frac{x^3}{3}+C} = Ae^{\frac{x^3}{3}},$$

i.e.,

$$y = Ae^{\frac{x^3}{3}} - 1$$

here  $A$  can be any nonzero real number. If  $y = -1$ , we have a constant solution  $y = -1$  which corresponds to  $A = 0$  in the above formula. So

$$y = Ae^{\frac{x^3}{3}} - 1,$$

where  $A$  can be any real number, gives the general solution. Now  $y(0) = 5$  implies  $A = 6$ , hence the solution to the initial value problem is

$$y = 6e^{\frac{x^3}{3}} - 1.$$

5. (10 points) Using Euler's method with step size  $h = \frac{1}{4}$  to approximate  $y\left(\frac{1}{2}\right)$  if

$$y' = 4x + y, y(0) = 8.$$

Solution: We have  $f(x, y) = 4x + y$  and

$$x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2},$$

so

$$y_1 = y_0 + f(x_0, y_0)h = 8 + (4 \times 0 + 8)\frac{1}{4} = 10,$$
$$y\left(\frac{1}{2}\right) \approx y_2 = y_1 + f(x_1, y_1)h = 10 + \left(4 \times \frac{1}{4} + 10\right)\frac{1}{4} = \frac{51}{4}.$$

6. (15 points) A parabolic tank with upper radius  $2m$  and height  $4m$  is full of water. Determine the work required to pump the water out of the tank. Setup the integral only, evaluation of the integral is not required. (Use the fact that the density of water is  $1000\text{kg}/\text{m}^3$  and the acceleration of

gravity is  $g = 10m/\text{sec}^2$ .)

Solution: The tank's shape is given by  $y = x^2$ . So

$$\begin{aligned} W &\approx \sum_{i=1}^n F_i d_i = \sum_{i=1}^n \rho V_i g d_i = \sum_{i=1}^n \rho (\pi x_i^2 \Delta y) g (4 - y_i) \\ &= \sum_{i=1}^n \rho (\pi y_i \Delta y) g (4 - y_i). \end{aligned}$$

So

$$\begin{aligned} W &= \int_0^4 \pi \rho g y (4 - y) dy = 10,000\pi \int_0^4 y (4 - y) dy \\ &= \frac{320,000}{3} \pi. \end{aligned}$$