First Midterm for Math 230
October 11, 2011

Last Name:_________________ First Name:_________________

1. Evaluate the indefinite integral

(a) (8 points) \[ \int \sin^2 x \cos^3 x \, dx; \]
Answer: Let \( u = \sin x \),
\[ \int \sin^2 x \cos^3 x \, dx = \int u^2 (1 - u^2) \, du = \frac{1}{3} u^3 - \frac{1}{5} u^5 + C \]
\[ = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C. \]

(b) (8 points) \[ \int \frac{x}{x^2 + 3x + 2} \, dx. \]
Answer:
\[ \frac{x}{x^2 + 3x + 2} = \frac{2}{x + 2} - \frac{1}{x + 1}, \]
hence
\[ \int \frac{x}{x^2 + 3x + 2} \, dx = 2 \ln |x + 2| - \ln |x + 1| + C. \]

2. Evaluate the definite integral

(a) (10 points) \[ \int_0^1 \frac{2x}{(1 + x^2)^9} \, dx; \]
Answer: Let \( u = 1 + x^2 \). We have
\[ \int_0^1 \frac{2x}{(1 + x^2)^9} \, dx = \int_1^2 u^{-9} \, du = \frac{1}{-8} u^{-8} \bigg|_1^2 = \frac{255}{2048}. \]

(b) (10 points) \[ \int_0^\infty xe^{-2x} \, dx. \]


Answer: This is an improper integral.

\[ \int_0^\infty xe^{-2x}dx \]

\[ = \lim_{t \to \infty} \int_0^t xe^{-2x}dx \]

\[ = \lim_{t \to \infty} \left( \frac{-xe^{-2x}}{2} \bigg|_0^t + \frac{1}{2} \int_0^t e^{-2x}dx \right) \]

\[ = \lim_{t \to \infty} \left( \frac{-te^{-2t}}{2} - \frac{1}{4}e^{-2t} + \frac{1}{4} \right) = \frac{1}{4}. \]

3. Solving the problems:

(a) (8 points) Find the area of the triangle \(ABC\) for \(A(3, 2, 1), B(2, 4, 3)\) and \(C(4, 3, 5)\).

Answer:

\[ \overrightarrow{AB} = (-1, 2, 2), \quad \overrightarrow{AC} = (1, 1, 4) \]

The area is equal to

\[ \frac{1}{2} | \overrightarrow{AB} \times \overrightarrow{AC} | = \frac{1}{2} |(6, 6, -3)| = \frac{9}{2}. \]

(b) (8 points) Find the volume of the parallelepiped formed by the vectors

\[ \vec{a} = (1, 2, 3), \quad \vec{b} = (2, 3, 4) \quad \text{and} \quad \vec{c} = (3, -1, 0). \]

Answer: The volume is

\[ \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & -1 & 0 \end{vmatrix} = |-5| = 5. \]

(c) (8 points) Find the distance from the point \(P(1, 2, -3)\) to the plane \(2x - y + 2z = 3\).

Answer: The distance

\[ D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2 \times 1 - 2 + 2 \times (-3) - 3|}{3} = 3. \]

(d) (8 points) Find an equation of the plane which contains a point \(P(1, 2, 3)\) and a straight line

\[ \frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z}{3}. \]
Answer: Select a point \( A(1, 2, 0) \) in the straight line. Let \( \vec{v} = \langle 1, 2, 3 \rangle \) be the direction vector of the straight line. One normal vector of the plane can be calculated as

\[
\vec{n} = \vec{AP} \times \vec{v} = \langle 0, 0, 3 \rangle \times \langle 1, 2, 3 \rangle = \langle -2, 1, 0 \rangle.
\]

Hence the plane equation using point \( P \) and normal \( \vec{n} \) is

\[-2(x - 1) + y - 2 = 0,
\]

i.e.,

\[2x - y = 0.\]

4. (12 points) Find the length of the curve

\[y = 1 + 2x^2, 0 \leq x \leq 4.\]

Answer: Length

\[
L = \int_0^4 \sqrt{1 + 4x} \, dx = \frac{2}{27} \left(37\sqrt{37} - 1\right).
\]

5. (10 points) Set up an integral representing the volume of the solid formed by rotating the region bounded by \( y = (x - 1)^2 \) and \( y = 2x + 1 \) about \( y \)-axis. **Setup the integral only, you are not required to evaluate the integral.**

Answer: The other intersection point is \((4, 9)\). Using cylindrical shell method, we have

\[
V = \int_0^4 2\pi x \left[ 2x + 1 - (x - 1)^2 \right] \, dx.
\]

6. (10 points) A parabolic tank with upper radius 2 m and height 4 m is full of water. Determine the work required to pump the water out of the tank. **Setup the integral only, evaluation of the integral is not required.** (Use the fact that the density of water is 1000 kg/m\(^3\) and the acceleration of gravity is \( g = 10 \text{ m/sec}^2 \).)

Answer:

\[
F = \int_0^4 \rho g A(y) h(y) \, dy
= 10^4 \int_0^4 \pi y(4 - y) \, dy.
\]