

First Midterm for Math 230

October 11, 2011

Last Name: \_\_\_\_\_ First Name: \_\_\_\_\_

1. Evaluate the indefinite integral

(a) (8 points)

$$\int \sin^2 x \cos^3 x dx;$$

Answer: Let  $u = \sin x$ ,

$$\begin{aligned} \int \sin^2 x \cos^3 x dx &= \int u^2 (1 - u^2) du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\ &= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C. \end{aligned}$$

(b) (8 points)

$$\int \frac{x}{x^2 + 3x + 2} dx.$$

Answer:

$$\frac{x}{x^2 + 3x + 2} = \frac{2}{x + 2} - \frac{1}{x + 1},$$

hence

$$\int \frac{x}{x^2 + 3x + 2} dx = 2 \ln |x + 2| - \ln |x + 1| + C.$$

2. Evaluate the definite integral

(a) (10 points)

$$\int_0^1 \frac{2x}{(1 + x^2)^9} dx;$$

Answer: Let  $u = 1 + x^2$ . We have

$$\int_0^1 \frac{2x}{(1 + x^2)^9} dx = \int_1^2 u^{-9} du = \frac{1}{-8} u^{-8} \Big|_1^2 = \frac{255}{2048}.$$

(b) (10 points)

$$\int_0^{\infty} x e^{-2x} dx.$$

Answer: This is an improper integral.

$$\begin{aligned} & \int_0^{\infty} x e^{-2x} dx \\ &= \lim_{t \rightarrow \infty} \int_0^t x e^{-2x} dx \\ &= \lim_{t \rightarrow \infty} \left( \frac{-x e^{-2x}}{2} \Big|_0^t + \frac{1}{2} \int_0^t e^{-2x} dx \right) \\ &= \lim_{t \rightarrow \infty} \left( \frac{-t e^{-2t}}{2} - \frac{1}{4} e^{-2t} + \frac{1}{4} \right) = \frac{1}{4}. \end{aligned}$$

3. Solving the problems:

- (a) (8 points) Find the area of the triangle  $ABC$  for  $A(3, 2, 1)$ ,  $B(2, 4, 3)$  and  $C(4, 3, 5)$ .

Answer:

$$\vec{AB} = \langle -1, 2, 2 \rangle, \vec{AC} = \langle 1, 1, 4 \rangle$$

The area is equal to

$$\frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |\langle 6, 6, -3 \rangle| = \frac{9}{2}.$$

- (b) (8 points) Find the volume of the parallelepiped formed by the vectors

$$\vec{a} = \langle 1, 2, 3 \rangle, \vec{b} = \langle 2, 3, 4 \rangle \text{ and } \vec{c} = \langle 3, -1, 0 \rangle.$$

Answer: The volume is

$$\left| \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & -1 & 0 \end{vmatrix} \right| = |-5| = 5.$$

- (c) (8 points) Find the distance from the point  $P(1, 2, -3)$  to the plane  $2x - y + 2z = 3$ .

Answer: The distance

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2 \times 1 - 2 + 2 \times (-3) - 3|}{3} = 3.$$

- (d) (8 points) Find an equation of the plane which contains a point  $P(1, 2, 3)$  and a straight line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z}{3}.$$

Answer: Select a point  $A(1, 2, 0)$  in the straight line. Let  $\vec{v} = \langle 1, 2, 3 \rangle$  be the direction vector of the straight line. One normal vector of the plane can be calculated as

$$\vec{n} = \overrightarrow{AP} \times \vec{v} = \langle 0, 0, 3 \rangle \times \langle 1, 2, 3 \rangle = \langle -2, 1, 0 \rangle.$$

Hence the plane equation using point  $P$  and normal  $\vec{n}$  is

$$-2(x - 1) + y - 2 = 0,$$

i.e.,

$$2x - y = 0.$$

4. (12 points) Find the length of the curve

$$y = 1 + 2x^{\frac{3}{2}}, 0 \leq x \leq 4.$$

Answer: Length

$$L = \int_0^4 \sqrt{1 + 9x} dx = \frac{2}{27} (37\sqrt{37} - 1).$$

5. (10 points) Set up an integral representing the volume of the solid formed by rotating the region bounded by  $y = (x - 1)^2$  and  $y = 2x + 1$  about  $y$ -axis. **Setup the integral only, you are not required to evaluate the integral.**

Answer: The other intersection point is  $(4, 9)$ . Using cylindrical shell method, we have

$$V = \int_0^4 2\pi x [2x + 1 - (x - 1)^2] dx.$$

6. (10 points) A parabolic tank with upper radius  $2m$  and height  $4m$  is full of water. Determine the work required to pump the water out of the tank. **Setup the integral only, evaluation of the integral is not required.** (Use the fact that the density of water is  $1000\text{kg}/\text{m}^3$  and the acceleration of gravity is  $g = 10\text{m}/\text{sec}^2$ .)

Answer:

$$\begin{aligned} F &= \int \rho g A(y) h(y) dy \\ &= 10^4 \int_0^4 \pi y (4 - y) dy. \end{aligned}$$