Foliations and Non Hausdorff Manifolds

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Outline

Foliations and Non Hausdorff Manifolds
  Plane Foliations
  Non Hausdorff Manifolds

The Topology of Non Hausdorff Manifolds
  Covering Properties
  Non–Rigid Manifolds
  Rigid Manifolds
  Realizing Groups

Applications to Foliations
  Automorphisms of Foliations
  A Rigid Foliation
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Systems of ODEs

\[
\frac{dx}{dt} = xy^2 - x^2 + y^3 - 1, \quad \frac{dy}{dt} = y^2 + xy.
\]
A **plane foliation** is a partition $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ of $\mathbb{R}^2$ by closed sets each homeomorphic to $\mathbb{R}$ such that:

there is an oriented atlas $\{(U_p, \phi_p) : p \in \mathbb{R}^2\}$ of $\mathbb{R}^2$ such that each component of $\phi(L_\alpha \cap U_p)$ is of the form $\{(x, y) \in \phi(U_p) : y = c\}$.

Each set $L_\alpha$ is called a **leaf** of the foliation.
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Leaf Spaces

The set of leaves forms a space called the leaf space; it carries the obvious quotient topology from $\mathbb{R}^2$.

The leaf space of a plane foliation is:
- a one–manifold, which is $T_1$, but rarely Hausdorff.
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Origin of Non Hausdorff Manifolds

Non Hausdorff manifolds arise as quotients of manifolds:

- Leaf space of foliations.
- Reduced twistor spaces in relativity theory.
- Models of space–time in ‘many-worlds’ interpretations of quantum mechanics.
Properties of Leaf Spaces

Lemma (Haefliger & Reeb)

Let $M$ be the leaf space of a plane foliation. Then $M$ is:

- an orientable, one–dimensional manifold,
- $T_1$ but rarely Hausdorff,
- second countable, and
- every point is a cut–point ($\iff$ simply connected).

Theorem (H&R)

Via the leaf space, there is (almost) a correspondence between plane foliations and orientable, second countable, simply connected one–manifolds.
An Opportunity...

- Plane foliations can be understood via non Hausdorff manifolds (the leaf spaces).
- The relevant manifolds are second countable but not metric.
- Classic concepts of analytic topology relevant:
  - covering properties,
  - generalized metric properties (developable, $G_\delta$ diagonal etc),
  - rigidity
  - etc.
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**Lindelof**

**Lemma**

*Let* $M$ *be a manifold. Then:*

-second countable $\iff$ Lindelof $\iff$ metaLindelof.*

**Corollary**

*Let* $M$ *be a manifold. Then:*

-paracompact $\implies$ metacompact $\implies$ second countable.*

-paracompact =

every open cover has a locally finite open refinement.*
Paracompact

Example

There is a metacompact manifold which is not paracompact.
Paracompact

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Metacompact

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There is a second countable manifold which is not metacompact.

\[ M = \mathbb{R} \cup (\mathbb{Q} \times \mathbb{N} \times [0, \infty)) : \]
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Homogeneity and Hausdorffness

A space $X$ is **homogeneous** iff

$$\forall x \neq y \in X \exists \text{ homeomorphism } h : X \to X : h(x) = y.$$ 

**Lemma**

*Hausdorff manifolds are homogeneous.*
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Hausdorffness and Rigidity

Theorem (cf Baillif & Gabard)
MetaLindelof homogeneous manifolds are Hausdorff.

So small non Hausdorff manifolds are not homogeneous...

Can they be rigid: have no non-trivial homeomorphisms?
Let $M$ be an $n$-manifold.

Define $R(M) = \text{all points of } M \text{ with a closed neighborhood homeomorphic to the closed } n\text{-ball.}$

**Lemma**

*If a manifold $M$, has $R(M) \neq \emptyset$, then $M$ is not rigid.*

**Theorem**

*Every metacompact manifold, $M$, has $R(M) \neq \emptyset$. Hence $M$ is not rigid.*
Rigid Manifolds Exist

**Theorem**

*There is a second countable rigid manifold.*

The manifold is:

- orientable, one-dimensional and simply connected.

**Theorem**

*There are arbitrarily large rigid manifolds.*

Uses: there are arbitrarily large cardinals $\lambda$ such that $\aleph_\lambda = \lambda$
Realizing Groups

**Theorem**

Every (countable) group can be realized as the autohomeomorphism group of a (second countable) manifold.
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Inner and Outer Automorphisms

An automorphism of a plane foliation $\mathcal{F}$ is an orientation preserving autohomeomorphism of the plane taking leaves to leaves.

$\text{Aut}(\mathcal{F}) =$ all automorphisms of $\mathcal{F}$.

An automorphism is inner if it carries each leaf to itself.

$\text{Inn}(\mathcal{F}) =$ all inner automorphisms. $\text{Out}(\mathcal{F}) = \text{Aut}(\mathcal{F})/\text{Inn}(\mathcal{F})$.

Lemma

$\text{Inn}(\mathcal{F})$ contains an uncountable free subgroup.

Call a plane foliation rigid iff $\text{Out}(\mathcal{F})$ is trivial.

Lemma

A foliation with rigid leaf space is rigid.
A Rigid Foliation – Ingredients:

A copy of \((0, 1)^2\) foliated by the vertical lines;
\(\forall n \geq 1\), \(n\) copies of \([0, 1) \times (0, 1)\) foliated by vertical lines;
and a ‘nice’ enumeration of a dense countable family of leaves.
A Rigid Foliation – Insert Buckles:

\[ \mathcal{F}_0 = \text{the copy of } (0,1)^2 \text{ foliated by the vertical lines.} \]
At stage \( n \) of the construction the \( n \) ‘buckles’ are inserted into a tubular neighborhood of the \( n \)th dense leaf.
A Rigid Foliation – Stage $n$:

This gives foliation $\mathcal{F}_n$. In the limit we get our rigid foliation.
A Rigid Foliation – Why Rigid?

Roughly speaking:

- Different dense leaves have different families of buckles in their tubular neighborhoods.
- Hence the dense set of leaves ‘look different’ from each other.
- Since the dense leaves are dense...
  it follows that all leaves ‘look different’ from each other.
- So any automorphism can only carry each leaf to itself — in other words the foliation is rigid.