1. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

(a) (15 points) \( x y' = (1 + 2x^2)y, \quad y(-1) = e, \) where \( y' = \frac{dy}{dx}. \)

**Solution:** It is a separable equation

\[
x \frac{dy}{dx} = (1 + 2x^2)y, \quad \frac{dy}{y} = \frac{1 + 2x^2}{x} dx,
\]

\[
\ln |y| = \ln |x| + x^2 + C, \quad |y| = |x|e^{x^2}e^C, \quad y = Axe^{x^2}, \quad A = \pm e^C.
\]

\[y(-1) = A(-1)e = e, \quad A = -1, \quad y(x) = -xe^{x^2}.
\]

(b) (15 points) \((t^2 + 2)x' + 4tx = 3, \quad x(0) = -1.

**Solution:** Divide both sides by \(t^2 + 2\) to get a first order linear differential equation:

\[
x' + \frac{4t}{t^2 + 2} x = \frac{3}{t^2 + 2}.
\]

The integrating factor is \(I(t) = e^{\int \frac{4t}{t^2 + 2} dt} = e^{2\ln(t^2 + 2)} = e^{\ln((t^2 + 2)^2)} = (t^2 + 2)^2.\) Then

\[
(t^2 + 2)^2 \left( x' + \frac{4t}{t^2 + 2} x \right) = (t^2 + 2)^2 \cdot \frac{3}{t^2 + 2}, \quad (t^2 + 2)^2 x' + 4t(t^2 + 2)x = 3t^2 + 6,
\]

\[
((t^2 + 2)^2 x)' = 3t^2 + 6, \quad (t^2 + 2)^2 x = \int (3t^2 + 6) dt = t^3 + 6t + C, \quad x = \frac{t^3 + 6t + C}{(t^2 + 2)^2}.
\]

\[x(0) = \frac{C}{4} = -1, \quad C = -4, \quad x(t) = \frac{t^3 + 6t - 4}{(t^2 + 2)^2}.
\]

2. (15 points) Suppose you drop a ball from the top of a building with the initial velocity 0 m/sec. The ball has mass of 0.2 kg. The air resistance force is given by \(R(v) = -\frac{v}{5}.\)

How long will it take the ball to reach one-half of its terminal velocity? Leave answer in exact form.

**Solution:** Here \(r = \frac{1}{5}\) and \(m = 0.2 = \frac{1}{5}\). Then \(\frac{r}{m} = 1.\)

The equation of the motion is either \(v' = -g - v\) when the \(x\)-axis is directed up or \(v' = g - v\) when the \(x\)-axis is directed down. In both cases the equation is separable or the first order linear. For \(v' = -g - v\) the solution is \(v(t) = Ce^{-t} - g.\)
The initial condition gives $C = g$. So, $v(t) = g(e^{-t} - 1)$. The terminal velocity is $\lim_{t \to \infty} v(t) = -g$.

We have to find time $t$ when $v(t) = -\frac{1}{2} \cdot g$. Then,

$$g(e^{-t} - 1) = -\frac{1}{2} \cdot g, \quad e^{-t} - 1 = -\frac{1}{2}, \quad e^{-t} = \frac{1}{2}, \quad -t = -\ln 2, \quad t = \ln 2 \text{ sec}$$

The answer is the same if you used the equation $v' = g - v$.

3. (15 points) Suppose the electrical circuit has a resistor of $R = 0.4 \, \Omega$ and an inductor of $L = 0.2 \, H$. Assume the voltage source is a constant $E = 0.6 \, V$. If the initial current is $0 \, A$ find the resulting current as a function of time. Simplify your answer and leave it in exact form.

Solution: The model is described by the IVP: $RI + L \frac{dI}{dt} = E, \quad I(0) = 0$.

When we plug in all constants the equation becomes $0.4I + 0.2 \frac{dI}{dt} = 0.6$ or,

after multiplication by 5, $\frac{dI}{dt} + 2I = 3$.

The equation is both the first order linear and separable. Let’s solve it as a separable equation:

$$\frac{dI}{3 - 2I} = dt, \quad \frac{1}{2} \ln|3 - 2I| = t + C_1, \quad \ln|3 - 2I| = -2t + C, \quad C = -2C_1.$$

Hence, $3 - 2I = Ae^{-2t}$ and $I(t) = \frac{3}{2} - \frac{1}{2} Ae^{-2t}$.

The initial condition gives $I(0) = \frac{3}{2} - \frac{1}{2} A = 0$. Then $A = 3$.

The resulting current is $I(t) = \frac{3}{2} - \frac{3}{2}e^{-2t} = 1.5 - 1.5e^{-2t}$

(If you treat the equation as the first order linear, then the integrating factor is $u = e^{2t}$)

4. (20 points) Using the method of undetermined coefficients find the general solution of the equation $y'' - y' - 2y = 2e^{2t}$

Show all the work. Mention a type of the equation.

Solution: It is a second-order linear non-homogeneous differential equation.

The characteristic equation of the corresponding homogeneous DE $r^2 - r - 2 = 0$ has two real roots $r = -1$ and $r = 2$.

The solution of the homogeneous DE is $y_h(t) = C_1 e^{-t} + C_2 e^{2t}$

The right hand side of the given non-homogeneous DE coincides with one of the homogeneous
solutions. Therefore to find a particular solution we have to use the form
\[ y_p = a te^{2t} \] (not \( y_p = a e^{2t} \)).

\[ y'_p = ae^{2t} + 2ae^{2t}, \quad y''_p = 4ae^{2t} + 4ate^{2t} \]

The left hand side of the given non-homogeneous DE is
\[ y'' - y' - 2y = 4ae^{2t} + 4ate^{2t} - ae^{2t} - 2ate^{2t} - 2ate^{2t} = 3ae^{2t} \]

It has to be equal the right hand side \( 2e^{2t} \). That gives \( a = \frac{2}{3} \) and \( y_p(t) = \frac{2}{3} te^{2t} \).

The general solution is \( y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{2t} + \frac{2}{3} te^{2t} \).

5. (20 points) Using variation of parameters technique find a particular solution to the equation
\[ y'' - 4y = e^{3t} \]

Show all the work. Mention a type of the equation.

**Solution:** It is a second-order linear non-homogeneous differential equation.

The characteristic equation of the corresponding homogeneous DE
\[ r^2 - 4 = 0 \] has two real roots \( r = -2 \) and \( r = 2 \).

The fundamental set of solutions is \( y_1 = e^{-2t}, \quad y_2 = e^{2t} \)

The Wronskian \( W = y_1y'_2 - y'_1y_2 = e^{-2t} \cdot 2e^{2t} - (-2)e^{-2t}e^{2t} = 2 + 2 = 4 \)

\[ v_1 = \int -y_2e^{3t} \frac{dt}{W} = -\frac{1}{4} \int e^{2t}e^{3t} dt = -\frac{1}{4} \int e^{5t} dt = -\frac{1}{20} e^{5t} \]

\[ v_2 = \int y_1e^{3t} \frac{dt}{W} = \frac{1}{4} \int e^{-2t}e^{3t} dt = \frac{1}{4} \int e^t dt = \frac{1}{4} e^t \]

A particular solution is \( y_p = v_1y_1 + v_2y_2 = -\frac{1}{20} e^{5t}e^{-2t} + \frac{1}{4} e^t e^{2t} = \frac{1}{5} e^{3t} \).

The general solution is \( y(t) = C_1y_1 + C_2y_2 + y_p = C_1e^{-t} + C_2e^{2t} + \frac{1}{5} e^{3t} \).

(If your fundamental set of solutions is \( y_1 = e^{2t}, \quad y_2 = e^{-2t} \) then \( W = -4 \) and \( v_1 = \frac{1}{4} e^t \),
\[ v_2 = -\frac{1}{20} e^{5t} \))

**Bonus problem** (15 points extra) Find the general solution of the equation \( y' = (y + t)^2 \).

Hint: Use the substitution \( x = y + t \)

**Solution:** \( x = y + t, \quad y = x - t, \quad y' = x' - 1 \)

So, the given equation becomes
$x' - 1 = x^2, \quad \frac{dx}{dt} = x^2 + 1, \quad \int \frac{dx}{x^2 + 1} = \int dt, \quad \tan^{-1} x = t + C, \quad x = \tan(t + C)$. Then $y = \tan(t + C) - t$. 