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LOW MACH NUMBER LIMIT OF VISCOUS COMPRESSIBLE MAGNETOHYDRODYNAMIC FLOWS*

XIANPENG HU† AND DEHUA WANG†

Abstract. The relationship between the compressible magnetohydrodynamic flows with low Mach number and the incompressible magnetohydrodynamic flows is investigated. More precisely, the convergence of weak solutions of the compressible isentropic viscous magnetohydrodynamic equations to the weak solutions of the incompressible viscous magnetohydrodynamic equations is proved as the density becomes constant and the Mach number goes to zero; that is, the corresponding incompressible limits are justified when the spatial domain is a periodic domain, the whole space, or a bounded domain.

Key words. compressible magnetohydrodynamic equations, isentropic, weak solutions, incompressible magnetohydrodynamic equations, low Mach number

AMS subject classifications. 35Q36, 35D05, 76W05

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1. Introduction.

Studies on magnetohydrodynamic flows always involve a choice at the onset to describe the system entirely in the context of either incompressible magnetohydrodynamics (MHD) or compressible MHD. For example, theoretic studies on turbulence have a particular leaning toward the incompressible model. This preference has largely been based on the benefits and advantages of the similarity of incompressible MHD to its hydrodynamic counterparts, and the practical consideration of limited computational resources. However, when the density of a flow is no longer invariant, the flow becomes much more complicated not only from the physical viewpoint but also from the mathematical consideration; see [3, 17, 18, 21, 22] and the references therein. Thus, it is a natural problem to consider the relation between incompressible MHD and compressible MHD. The equations of the isentropic compressible viscous magnetohydrodynamic flows in N spatial dimensions have the following form [3, 21, 22]:

$$\begin{cases} \widetilde{\rho}_t + \operatorname{div}(\widetilde{\rho}\,\widetilde{\mathbf{u}}) = 0, \\ (\widetilde{\rho}\,\widetilde{\mathbf{u}})_t + \operatorname{div}(\widetilde{\rho}\,\widetilde{\mathbf{u}} \otimes \widetilde{\mathbf{u}}) + \nabla \widetilde{p}(\widetilde{\rho}) = (\nabla \times \widetilde{\mathbf{H}}) \times \widetilde{\mathbf{H}} + \widetilde{\mu}\Delta\widetilde{\mathbf{u}} + \widetilde{\lambda}\nabla\operatorname{div}\widetilde{\mathbf{u}}, \\ \widetilde{\mathbf{H}}_t - \nabla \times (\widetilde{\mathbf{u}} \times \widetilde{\mathbf{H}}) = -\nabla \times (\widetilde{\nu}\,\nabla \times \widetilde{\mathbf{H}}), \quad \operatorname{div}\widetilde{\mathbf{H}} = 0, \end{cases} \tag{1.1}$$

where $\widetilde{\mu} > 0$ is the shear viscosity, $\widetilde{\lambda}$ is the bulk viscosity satisfying $2\widetilde{\mu} + N\widetilde{\lambda} > 0$, and $\widetilde{\nu} > 0$ is the magnetic viscosity; and $\widetilde{\rho}$ denotes the density, $\widetilde{\mathbf{u}} \in \mathbb{R}^N$ the velocity, $\widetilde{\mathbf{H}} \in \mathbb{R}^N$ the magnetic field, and $\widetilde{p}(\widetilde{\rho}) = a\widetilde{\rho}^\gamma$ the pressure with constant $a > 0$ and the adiabatic exponent $\gamma > 1$. The symbol $\otimes$ denotes the Kronecker tensor product. The first equation in (1.1) is called the continuity equation, and the third equation in (1.1) is called the induction equation.

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†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260 (xih15@pitt.edu, dwang@math.pitt.edu). The first author's research was supported in part by the National Science Foundation under grant DMS-0604362 and by the Mellon Predoctoral Fellowship of the University of Pittsburgh. The second author's research was supported in part by the National Science Foundation under grant DMS-0604362 and by the Office of Naval Research under grant N00014-07-1-0668.

From the physics point of view, the compressible flow behaves asymptotically like an incompressible flow when the density is almost constant, and the velocity and the magnetic field are small, in a large time scale. More precisely, we scale $\widetilde{\rho}$, $\widetilde{\mathbf{u}}$, and $\widetilde{\mathbf{H}}$ in the following way:

$$\widetilde{\rho} = \rho(x, \varepsilon t), \quad \widetilde{\mathbf{u}} = \varepsilon \mathbf{u}(x, \varepsilon t), \quad \widetilde{\mathbf{H}} = \varepsilon \mathbf{H}(x, \varepsilon t), \tag{1.2}$$

and we assume that the coefficients $\widetilde{\mu}$, $\widetilde{\lambda}$, and $\widetilde{\nu}$ are small and scaled as

$$\widetilde{\mu} = \varepsilon \mu_\varepsilon, \quad \widetilde{\lambda} = \varepsilon \lambda_\varepsilon, \quad \widetilde{\nu} = \varepsilon \nu_\varepsilon, \tag{1.3}$$

where $\varepsilon \in (0,1)$ is a small parameter and the normalized coefficients μ_ε, λ_ε, and ν_ε satisfy

$$\mu_\varepsilon \to \mu, \quad \lambda_\varepsilon \to \lambda, \quad \nu_\varepsilon \to \nu \quad \text{as } \varepsilon \to 0+, \tag{1.4}$$

with $\mu > 0$, $2\mu + N\lambda > 0$, and $\nu > 0$. Such a scaling as (1.3) ensures that the limit system as $\varepsilon \to 0$ is not of an Euler type. Also notice that the parameter ε in the front of the magnetic field $\mathbf{H}$ in (1.2) can be understood as the reciprocal of Alfv́en number [27]. Under those scalings, system (1.1) yields

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu_\varepsilon \Delta \mathbf{u} - \lambda_\varepsilon \nabla \operatorname{div} \mathbf{u} + \frac{a}{\varepsilon^2} \nabla \rho^\gamma = (\nabla \times \mathbf{H}) \times \mathbf{H}, \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu_\varepsilon \nabla \times \mathbf{H}), \quad \operatorname{div} \mathbf{H} = 0. \end{cases} \tag{1.5}$$

The existence of global weak solutions to (1.5) has been investigated in Hu and Wang [17] (and in Hu and Wang [18] for the nonisentropic case). From the mathematical point of view, it is reasonable to expect that, as $\rho \to 1$, the first equation in (1.5) yields the limit $\operatorname{div}\mathbf{u} = 0$, which is the incompressible condition of a fluid, and the first two terms in the second equation of (1.5) become

$$\mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

On the other hand, the incompressible MHD equations read

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H}, \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \\ \operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \mathbf{H} = 0. \end{cases} \tag{1.6}$$

Thus, roughly speaking, it is also reasonable to expect from the mathematical point of view that weak solutions of (1.5) converge in certain suitable functional spaces to the weak solutions of (1.6) as ρ goes to a constant such as 1 and ε goes to 0, and the hydrostatic pressure p in (1.6) is the "limit" of $(\rho^\gamma - 1)/\varepsilon^2$ in (1.5). This paper is devoted to the rigorous justification of the convergence of that incompressible limit (i.e., the low Mach number limit) for global weak solutions of the compressible isentropic MHD equations.

In this paper, we shall establish the incompressible limit of (1.5) in three types of spatial domains: the torus $\mathbb{T}$ (in this case, all the functions are defined on $\mathbb{R}^N$ and assumed to be periodic with period 2π for all directions, that is, $\mathbb{T} = [0, 2\pi]^N$), the whole space $\mathbb{R}^N$, and a sufficiently smooth bounded domain $\Omega \in \mathbb{R}^N$, $N = 2, 3$. The study in the bounded smooth domain with no-slip boundary condition on the velocity is much harder than that in the other two cases, because in bounded domains, there are extra difficulties arising from the appearance of the boundary layers, and the subtle interactions between dissipative effects and wave propagation near the boundary, and hence requires a different approach. We remark that the incompressible limits for compressible isentropic Navier–Stokes equations have been investigated in [25] for the whole space $\mathbb{R}^N$ and the periodic domain using the group method, and in [6] for a bounded domain. These results have been extended by others, such as the authors of [2, 9, 5, 26, 34]. We also notice that in [15] convergence results were proved for well-prepared data as long as the solution of incompressible limit is suitably smooth. For the case of nonisentropic flows, see [12, 13] for some recent studies. For other related studies on the incompressible limits of viscous and inviscid flows, see [1, 4, 11, 16, 20, 19, 23, 29, 28, 31, 32] and the references in [12]. Comparing with those works on the compressible Navier–Stokes equations, we will encounter extra difficulties in studying the compressible MHD equations. More precisely, besides the possible oscillation of the density, the appearance of the boundary layer, and the interactions between dissipative effects and wave propagation, the appearance of the magnetic field and the coupling effect between the hydrodynamic motion and the magnetic field should also be taken into considerations with new estimates. We will overcome all these difficulties by using the group method, Strichartz's estimate, and the weak convergence method to establish the convergence of weak solutions of the compressible isentropic MHD equations (1.5) to weak solutions of the incompressible MHD equations (1.6) as the density goes to a constant and ε goes to 0 in the periodic case and the whole space case. More precisely, we will show that, for any fixed $T > 0$, in the periodic case, the incompressible part of the velocity converges strongly to a divergence-free vector field in $L^2([0,T], L^2(\mathbb{T}))$, while the gradient part of the velocity converges weakly to zero; and in the whole space case, due to Strichartz's estimate, the gradient part of the velocity converges strongly to 0 in $L^2([0,T], L^2_{loc}(\mathbb{R}^{\mathbb{N}}))$, while the strong convergence of the incompressible part of the velocity holds only in the local sense. However, this method does not apply to the case of bounded domains because of subtle interactions between dissipative effects and wave propagation near the boundary. Instead, we will use the spectral analysis of the semigroup generated by the dissipative wave operator, together with Duhamel's principle. Finally, we remark that the incompressible flow can also be derived from the vanishing Debye length type limit of a compressible flow with a Poisson damping. We refer the interested readers to [7, 8].

We organize the rest of the paper as follows. In section 2, we will give the setting of our problem and state our main results. In section 3, we discuss the convergence of the incompressible limit in the periodic case. In section 4, we will investigate the convergence of the incompressible limit in the whole space $\mathbb{R}^N$. Finally, in section 5, we will study the convergence of the incompressible limit in the bounded domain.

2. Main results.

In this section, we describe the setting of our problem and state our main results. First, we denote by P the orthogonal projection onto incompressible vector fields, i.e.,

$$v = Pv + Qv, \quad \text{with} \quad \operatorname{div}(Pv) = 0,\ \operatorname{curl}(Qv) = 0,$$

for all $v \in L^2$. Indeed, in view of results in [14], we know that the operators P and Q are linear bounded operators in $W^{s,p}$ for all $s \geq 0$ and $1 < p < \infty$ in the whole space or bounded domains with smooth boundaries. Second, let us explain the notation of weak solutions to the incompressible MHD equations as follows: Given the initial conditions $\mathbf{u}_0 \in L^2$, $\mathbf{H}_0 \in L^2$ such that $\mathrm{div}\mathbf{u}_0 = 0$ and $\mathrm{div}\mathbf{H}_0 = 0$, $(\mathbf{u}, \mathbf{H})$ is a weak solution of (1.6) satisfying

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0, \tag{2.1}$$

where

$$\mathbf{u} \in C([0,T]; L^2_{weak}) \cap L^2([0,T]; H^1(\Omega)),\ \mathbf{H} \in C([0,T]; L^2_{weak}) \cap L^2([0,T]; H^1(\Omega))$$

if for all $T < \infty$, $\psi \in C_0^\infty(\Omega)$ with $\mathrm{div}\psi = 0$, and $\varphi \in C_0^\infty([0,T))$ we have

$$\begin{aligned}&\psi(0)\int_\Omega \mathbf{u}_0\varphi dx + \int_0^t \psi'(t)\int_\Omega \mathbf{u}\cdot\varphi dxdt + \int_0^t \psi(t)\int_\Omega (\mathbf{u}_i\partial_i\varphi_j\mathbf{u}_j - \mu\nabla\mathbf{u} : \nabla\varphi)\, dxdt\\ &= -\int_0^t\int_\Omega \psi(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\varphi dxdt\end{aligned}$$

and

$$\begin{aligned}&\psi(0)\int_\Omega \mathbf{H}_0\varphi dx + \int_0^t \psi'(t)\int_\Omega \mathbf{H}\cdot\varphi dxdt + \int_0^t \psi(t)\int_\Omega (\mathbf{u}\times\mathbf{H})\cdot(\nabla\times\varphi)dxdt\\ &= \nu\int_0^t \psi(t)\int_\Omega (\nabla\times\mathbf{H})\cdot(\nabla\times\varphi)dxdt.\end{aligned}$$

For more details as to the existence and regularity of weak solutions to the incompressible MHD equations, we refer the readers to [10, 33]. Now, we can state our main results case by case.

2.1. The periodic case. Let us begin with the periodic case. We consider a sequence of global weak solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)$ of the compressible MHD equations (1.5) in $\mathbb{T}$ and assume that

$$\rho_\varepsilon \in L^\infty([0,T]; L^\gamma(\mathbb{T})), \quad \mathbf{u}_\varepsilon \in L^2([0,T]; H^1(\mathbb{T})),$$

$$\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 \in L^\infty([0,T]; L^1(\mathbb{T})), \quad \rho_\varepsilon\mathbf{u}_\varepsilon \in C\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}\right),$$

$$\mathbf{H}_\varepsilon \in L^2([0,T]; H^1(\mathbb{T})) \cap C([0,T]; L^2_{weak}(\mathbb{T}))$$

for all $T \in (0,\infty)$, where $C([0,T]; L^p_{weak})$ denotes the functions which are continuous with respect to $t \in [0,T]$ with values in L^p endowed with the weak topology. We require (1.5) to hold in the sense of distributions. Finally, we prescribe initial conditions

$$\rho_\varepsilon|_{t=0} = \rho_\varepsilon^0, \quad \rho_\varepsilon\mathbf{u}_\varepsilon|_{t=0} = m_\varepsilon^0 = \rho_\varepsilon^0\mathbf{u}_\varepsilon^0, \quad \mathbf{H}_\varepsilon|_{t=0} = \mathbf{H}_\varepsilon^0, \tag{2.2}$$

where $\rho_\varepsilon^0 \geq 0$, $\rho_\varepsilon^0 \in L^\gamma(\mathbb{T})$, $m_\varepsilon^0 \in L^{2\gamma/(\gamma+1)}(\mathbb{T})$, $m_\varepsilon^0 = 0$ on $\{\rho_\varepsilon^0 = 0\}$, $\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 \in L^1(\mathbb{T})$, and $\mathbf{H}_\varepsilon^0 \in L^2(\mathbb{T})$. Furthermore, we assume that $\sqrt{\rho_\varepsilon^0}\mathbf{u}_\varepsilon^0$ and $\mathbf{H}_\varepsilon^0$ converge weakly in L^2 to $\mathbf{u}_0$ and $\mathbf{H}_0$, respectively, and that we have

(2.3)

$$\frac{1}{2}\int_\mathbb{T}\left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 + |\mathbf{H}_\varepsilon^0|^2\right) dx + \frac{a}{\varepsilon^2(\gamma-1)}\int_\mathbb{T}\left((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0(\overline{\rho_\varepsilon})^{\gamma-1} + (\gamma-1)(\overline{\rho_\varepsilon})^\gamma\right) \leq C,$$

$$\overline{\rho_\varepsilon} = (2\pi)^{-N}\int_\mathbb{T}\rho_\varepsilon^0 dx \to 1 \quad \text{as } \varepsilon \to 0,$$

where here and hereafter C denotes a generic positive constant independent of ε. Notice that (2.3) implies that, roughly speaking, ρ_ε^0 is of order $\overline{\rho_\varepsilon} + O(\varepsilon)$. We assume finally that the total energy is conserved in the sense that

$$E_\varepsilon(t) + \int_0^t D_\varepsilon(s)ds \le E_\varepsilon^0 \quad \text{a.e} \quad t \in [0,T], \tag{2.4}$$

where

$$E_\varepsilon = \frac{1}{2}\int_\Omega \left(\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + |\mathbf{H}_\varepsilon|^2 + \frac{a}{\varepsilon^2(\gamma-1)}\rho_\varepsilon^\gamma\right) dx,$$

$$D_\varepsilon = \int_\Omega \left(\mu_\varepsilon|D\mathbf{u}_\varepsilon|^2 + \lambda_\varepsilon(\mathrm{div}\mathbf{u}_\varepsilon)^2 + \nu_\varepsilon|\nabla\times\mathbf{H}_\varepsilon|^2\right) dx,$$

and

$$E_\varepsilon^0 = \frac{1}{2}\int_\Omega \left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 + |\mathbf{H}_\varepsilon^0|^2 + \frac{a}{\varepsilon^2(\gamma-1)}(\rho_\varepsilon^0)^\gamma\right) dx,$$

where Ω is equal to $\mathbb{T}$ in the periodic case, and later is the whole space or a bounded domain.

We now recall the results in [17] which yield the existence of such a solution with the above properties precisely as $\gamma > \frac{N}{2}$ for $N = 2, 3$. We state the following theorem.

Theorem 2.1 (the periodic case). *Assume that* $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ *is a sequence of weak solutions to the compressible MHD equations* (1.5) *in the periodic domain* $\mathbb{T}$ *with initial data* $\{(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0, \mathbf{H}_\varepsilon^0)\}_{\varepsilon>0}$*, satisfying the conditions* (2.2)–(2.4) *and* $\gamma > \frac{N}{2}$, $N = 2, 3$. *Also assume that* $(\mathbf{u}, \mathbf{H}) \in [L^2([0,T]; H^1(\mathbb{T})) \cap L^\infty([0,T]; L^2(\mathbb{T}))]^2$ *is a weak solution to the incompressible MHD equations* (1.6) *with initial data* $\mathbf{u}|_{t=0} = P\mathbf{u}_0$ *and* $\mathbf{H}|_{t=0} = \mathbf{H}_0$. *Then, for any finite number* T*, up to a subsequence, the global weak solutions* $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ *converge to* $(\mathbf{u}, \mathbf{H})$. *More precisely, as* $\varepsilon \to 0$,

$$\rho_\varepsilon \text{ converges to } 1 \text{ in } C([0,T]; L^\gamma(\Omega));$$

$$P\mathbf{u}_\varepsilon \text{ converges strongly to } \mathbf{u} \text{ in } L^2([0,T]; L^p(\mathbb{T})) \text{ for all } 1 \le p < \frac{2N}{N-2};$$

$$Q\mathbf{u}_\varepsilon \text{ converges weakly to } 0 \text{ in } L^2([0,T]; H^1(\mathbb{T}));$$

$$\mathbf{H}_\varepsilon \text{ converges to } \mathbf{H} \text{ strongly in } L^2([0,T]; L^2(\mathbb{T})) \text{ and weakly in } L^2([0,T]; H^1(\mathbb{T})),$$

where, for convenience, we will denote ∞ *by* $\frac{2N}{N-2}$ *if* $N = 2$ *in this paper.*

2.2. The whole space case. Next, we turn to the whole space case. For the convenience of presentation, we discuss only the case when $a = 1$. In order to define weak solutions in the whole space, the following special type of Orlicz spaces $L_q^p(\Omega)$ is needed (see Appendix A in [24]):

$$L_q^p(\Omega) = \left\{f \in L_{loc}^1(\Omega) : \ f\chi_{\{|f|<\eta\}} \in L^q(\Omega), \ f\chi_{\{|f|\ge\eta\}} \in L^p(\Omega) \text{ for some } \eta > 0\right\},$$

where χ denotes the characteristic function of a set. We consider a sequence of weak solutions $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ in the whole space $\mathbb{R}^N$ with initial data $\{(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0, \mathbf{H}_\varepsilon^0)\}_{\varepsilon>0}$, satisfying the same conditions (2.2) and (2.4) as in the periodic case. In addition, the weak solutions $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ satisfy the following conditions at infinity:

$$\rho_\varepsilon \to 1, \qquad \mathbf{u}_\varepsilon \to 0, \qquad \mathbf{H}_\varepsilon \to 0 \quad \text{as } |x| \to \infty$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 + |\mathbf{H}_\varepsilon^0|^2\right) dx + \frac{a}{\varepsilon^2(\gamma-1)}\int_{\mathbb{R}^N} \left((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + (\gamma-1)\right) dx \le C. \tag{2.5}$$

As pointed out in [17], one can show that, for any fixed $\varepsilon > 0$, there exists a global weak solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)$ to the compressible MHD equations (1.5) defined by

$$\rho_\varepsilon - 1 \in L^\infty([0,T]; L_2^\gamma(\mathbb{R}^N)),$$

$$\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \in L^\infty([0,T]; L^2(\mathbb{R}^N)),$$

$$\nabla \mathbf{u}_\varepsilon \in L^2([0,T]; L^2(\mathbb{R}^N)),$$

$$\mathbf{H}_\varepsilon \in L^2([0,T]; H^1(\mathbb{R}^N)) \cap L^\infty([0,T]; L^2(\mathbb{R}^N)),$$

satisfying, in addition,

$$\rho_\varepsilon\mathbf{u}_\varepsilon \in C([0,T]; L_{loc}^{2\gamma/(\gamma+1)}(\mathbb{R}^N)),$$

$$\rho_\varepsilon \in C([0,T]; L_{loc}^p(\mathbb{R}^N))$$

if $1 \le p < \gamma$ for all finite numbers T.

Now we are ready to state our result in the whole space as follows.

THEOREM 2.2 (the whole space case). *Assume that* $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ *is a sequence of weak solutions to the compressible MHD equations* (1.5) *in the whole space* $\mathbb{R}^N$ *with the initial data* $\{(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0, \mathbf{H}_\varepsilon^0)\}_{\varepsilon>0}$*, satisfying the conditions* (2.2)*,* (2.4)*,* (2.5) *and* $\gamma > \frac{N}{2}$*,* $N = 2, 3$*. Also assume that* $(\mathbf{u}, \mathbf{H}) \in [L^2([0,T]; H^1(\mathbb{R}^N)) \cap L^\infty([0,T]; L^2(\mathbb{R}^N))]^2$ *is a weak solution to the incompressible MHD equations* (1.6) *with initial data* $\mathbf{u}|_{t=0} = P\mathbf{u}_0$ *and* $\mathbf{H}|_{t=0} = \mathbf{H}_0$*. Then, for any finite number* T*, up to a subsequence, the global weak solutions* $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ *converge to* $(\mathbf{u}, \mathbf{H})$*. More precisely, as* $\varepsilon \to 0$*,*

$$\rho_\varepsilon \textit{ converges to } 1 \textit{ in } C([0,T]; L^\gamma(\Omega));$$

$$P\mathbf{u}_\varepsilon \textit{ converges strongly to } \mathbf{u} \textit{ in } L^2([0,T]; L_{loc}^p(\mathbb{R}^N)) \quad \textit{for all } 1 \le p < \frac{2N}{N-2};$$

$$Q\mathbf{u}_\varepsilon \textit{ converges strongly to } 0 \textit{ in } L^2([0,T]; L^q(\mathbb{R}^N)) \quad \textit{for all } 2 < q < \frac{2N}{N-2};$$

$$\mathbf{H}_\varepsilon \textit{ converges to } \mathbf{H} \textit{ strongly in } L^2([0,T]; L^2(\mathbb{R}^N)) \textit{ and weakly in } L^2([0,T]; H^1(\mathbb{R}^N)).$$

2.3. The bounded domain case. The third case we will address in this paper is the incompressible limit in a bounded domain Ω. For convenience of presentation, we also discuss only the situation when $a = 1$. In order to state precisely our main theorem, we first introduce a geometrical condition on Ω. Let us consider the following overdetermined problem:

$$-\Delta\psi = \lambda\psi \quad \text{in } \Omega, \qquad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \qquad \text{and } \psi \text{ is constant on } \partial\Omega. \tag{2.6}$$

A solution to (2.6) is said to be trivial if $\lambda = 0$ and ψ is a constant. We say that Ω satisfies the assumption (A) in [6] if all the solutions to (2.6) are trivial. In the two dimensional space, it is proved that every bounded, simply connected open set Ω with Lipschitz boundary satisfies (A).

We consider a sequence of weak solutions $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ in a bounded domain Ω with initial data $\{(\rho^0_\varepsilon, \mathbf{u}^0_\varepsilon, \mathbf{H}^0_\varepsilon)\}_{\varepsilon>0}$ and boundary condition

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = 0, \quad \mathbf{H}_\varepsilon|_{\partial\Omega} = 0, \tag{2.7}$$

satisfying the same conditions (2.2) and (2.4) as in the periodic case. And the initial data of the weak solutions $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ satisfy

$$\frac{1}{2}\int_\Omega \left(\rho^0_\varepsilon|\mathbf{u}^0_\varepsilon|^2 + |\mathbf{H}^0_\varepsilon|^2\right) dx + \frac{a}{\varepsilon^2(\gamma-1)}\int_\Omega \left((\rho^0_\varepsilon)^\gamma - \gamma\rho^0_\varepsilon + (\gamma-1)\right) dx \le C. \tag{2.8}$$

As shown in [17], for any fixed $\varepsilon > 0$, there exists a global weak solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)$ to the compressible MHD equations (1.5) defined by

$$\rho_\varepsilon \in L^\infty([0,T]; L^\gamma(\Omega)),$$

$$\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \in L^\infty([0,T]; L^2(\Omega)),$$

$$\nabla\mathbf{u}_\varepsilon \in L^2([0,T]; H^1(\Omega)),$$

$$\mathbf{H}_\varepsilon \in L^2([0,T]; H^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)),$$

satisfying, in addition,

$$\rho_\varepsilon\mathbf{u}_\varepsilon \in C([0,T]; L^{2\gamma/(\gamma+1)}(\Omega)),$$

$$\rho_\varepsilon \in C([0,T]; L^p_{loc}(\Omega))$$

if $1 \le p < \gamma$ for all finite numbers T.

Our main result in bounded domains reads as follows.

Theorem 2.3 (the bounded domain case). *Assume that* $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ *is a sequence of weak solutions to the compressible MHD equations* (1.5) *in a bounded domain* Ω *with initial data* $\{(\rho^0_\varepsilon, \mathbf{u}^0_\varepsilon, \mathbf{H}^0_\varepsilon)\}_{\varepsilon>0}$ *and boundary condition* (2.7)*, satisfying the conditions* (2.2)*,* (2.4)*,* (2.8) *and* $\gamma > \frac{N}{2}$*,* $N = 2, 3$*. Also assume that* $(\mathbf{u}, \mathbf{H}) \in [L^2([0,T]; H^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega))]^2$ *is a weak solution to the incompressible MHD equations* (1.6) *with initial data* $\mathbf{u}|_{t=0} = P\mathbf{u}_0$ *and* $\mathbf{H}|_{t=0} = \mathbf{H}_0$ *and boundary conditions* $\mathbf{u}|_{\partial\Omega} = 0$ *and* $\mathbf{H}|_{\partial\Omega} = 0$*. Then, for any finite number* T*, as* ε *goes* *to* 0*, the global weak solutions* $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ *converge to* $(\mathbf{u}, \mathbf{H})$*. More precisely, as* $\varepsilon \to 0$*,*

$$\rho_\varepsilon \textit{ converges to } 1 \textit{ in } C([0,T]; L^\gamma(\Omega));$$

$$\mathbf{u}_\varepsilon \textit{ converges to } \mathbf{u} \textit{ weakly in } L^2(\Omega \times (0,T)) \textit{ and strongly if } \Omega \textit{ satisfies } \text{(A)};$$

$$\mathbf{H}_\varepsilon \textit{ converges to } \mathbf{H} \textit{ strongly in } L^2([0,T]; L^2(\Omega)) \textit{ and weakly in } L^2([0,T]; H^1(\Omega)).$$

Remark 2.1. In fact, we will split the eigenvectors $\{\Psi_{k,0}\}_{k\in\mathbb{N}}$ of the Laplace equation with Neumann boundary condition into two classes: those which are not constant on $\partial\Omega$ will generate a boundary layer and will be quickly damped, thus converging strongly to 0; those which are constant on $\partial\Omega$, for which no boundary layer forms, will remain oscillating forever and lead only to weak convergence. Hence, if (A) is not satisfied, $\mathbf{u}_\varepsilon$ will in general converge only weakly and not strongly to $\mathbf{u}$. In particular, in the bounded, simply connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, the boundary layer will always be generated, and hence $\mathbf{u}_\varepsilon$ will converge strongly to zero.

3. The periodic case.

In this section, we will prove Theorem 2.1.

3.1. A priori bounds and consequences.

We first deduce from (2.4) and from the conservation of mass that we have for almost all $t \geq 0$

(3.1)
$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}} \left(\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + |\mathbf{H}_\varepsilon|^2 + \frac{a}{\varepsilon^2(\gamma-1)}\left(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon(\overline{\rho_\varepsilon})^{\gamma-1} + (\gamma-1)(\overline{\rho_\varepsilon})^\gamma\right)\right) dx \\
&\qquad + \int_0^t \int_{\mathbb{T}} \left(\mu_\varepsilon|D\mathbf{u}_\varepsilon|^2 + \lambda_\varepsilon(\operatorname{div}\mathbf{u}_\varepsilon)^2 + \nu_\varepsilon|\nabla\times\mathbf{H}_\varepsilon|^2\right) dxds \\
&\leq \frac{1}{2}\int_{\mathbb{T}} \left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 + |\mathbf{H}_\varepsilon^0|^2 + \frac{a}{\varepsilon^2(\gamma-1)}\left((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0(\overline{\rho_\varepsilon})^{\gamma-1} + (\gamma-1)(\overline{\rho_\varepsilon})^\gamma\right)\right) dx \leq C.
\end{aligned}$$

From (3.1) we see that $\rho_\varepsilon|\mathbf{u}_\varepsilon|^2$, $|\mathbf{H}_\varepsilon|^2$, and $\frac{1}{\varepsilon^2}\left(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon(\overline{\rho_\varepsilon})^{\gamma-1} + (\gamma-1)(\overline{\rho_\varepsilon})^\gamma\right)$ are bounded in $L^\infty([0,T]; L^1(\mathbb{T}))$, and $D\mathbf{u}_\varepsilon$ and $\nabla\times\mathbf{H}_\varepsilon$ are bounded in $L^2([0,T]; L^2(\mathbb{T}))$. In particular, we see that ρ_ε is bounded in $L^\infty([0,T]; L^\gamma(\mathbb{T}))$ for all $T \in (0,\infty)$ due to the fact that, for ε small enough, $\overline{\rho_\varepsilon} \in (\frac{1}{2}, \frac{3}{2})$ and thus, for all $\delta > 0$, there exists some $\eta > 0$ such that

$$x^\gamma + (\gamma-1)(\overline{\rho_\varepsilon})^\gamma - \gamma x(\overline{\rho_\varepsilon})^{\gamma-1} \geq \eta|x - \overline{\rho_\varepsilon}|^\gamma \quad \text{if} \quad |x - \overline{\rho_\varepsilon}| \geq \delta, \quad x \geq 0. \tag{3.2}$$

As in [25] $\mathbf{u}_\varepsilon$ is bounded in $L^2([0,T]; H^1(\mathbb{T}))$ for all $T \in (0,\infty)$. In fact, we deduce from Hölder's and Poincáre's inequalities that we have for all $T \in (0,\infty)$

$$\int_0^T \int_{\mathbb{T}} \rho_\varepsilon \left|\mathbf{u}_\varepsilon - (2\pi)^{-N}\int_{\mathbb{T}} \mathbf{u}_\varepsilon dx\right|^2 dxdt \leq C\|\rho_\varepsilon\|_{L^\infty([0,T];L^\gamma)}\|D\mathbf{u}_\varepsilon\|^2_{L^2([0,T];L^2)} \leq C;$$

hence, in view of the above bound on $\rho_\varepsilon|\mathbf{u}_\varepsilon|^2$, we get

$$C \geq \int_0^T \int_{\mathbb{T}} \rho_\varepsilon \left(\int_{\mathbb{T}} \mathbf{u}_\varepsilon dx\right)^2 dxdt = \left(\int_{\mathbb{T}} \rho_\varepsilon^0 dx\right) \int_0^T \left(\int_{\mathbb{T}} \mathbf{u}_\varepsilon dx\right)^2 dt.$$

Since (2.3) implies that ρ_ε^0 converges to 1 in measure, and hence, up to a subsequence, in $L^1(\mathbb{T})$, we can deduce a bound on $\mathbf{u}_\varepsilon$ in $L^2([0,T];L^2)$ by using Poincáre's inequality again. Indeed, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}}|\mathbf{u}_\varepsilon|^2dxdt &\le 2\int_0^T\int_{\mathbb{T}}\left|\mathbf{u}_\varepsilon-(2\pi)^{-N}\int_{\mathbb{T}}\mathbf{u}_\varepsilon dx\right|^2dxdt+2(2\pi)^{-N}\int_0^T\left|\int_{\mathbb{T}}\mathbf{u}_\varepsilon\right|^2dxdt\\ &\le C\left(1+\int_0^T\int_{\mathbb{T}}|\nabla\mathbf{u}_\varepsilon|^2dxdt\right)\le C.\end{aligned}$$

From now on, we assume that, up to a subsequence, $\mathbf{u}_\varepsilon$ converges weakly to some $\mathbf{u}$ in $L^2([0,T];H^1(\mathbb{T}))$ for all $T>0$. On the other hand, the bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T];L^2(\mathbb{T}))$ and the bound on $\nabla\mathbf{H}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{T}))$, combining the Gagliardo–Nirenberg inequality

$$\|\mathbf{u}\|_{L^{\frac{8}{3}}([0,T];L^4(\mathbb{T}))}\le\|\mathbf{u}\|^{\frac14}_{L^\infty([0,T];L^2(\mathbb{T}))}\|\nabla\mathbf{u}\|^{\frac34}_{L^2([0,T];L^2(\mathbb{T}))},$$

imply that $\mathbf{H}_\varepsilon$ is bounded in $L^{8/3}([0,T];L^4(\mathbb{T}))$, and we can also assume that $\mathbf{H}_\varepsilon$ converges weakly to some $\mathbf{H}$ in $L^2([0,T];H^1(\mathbb{T}))$ with $\mathrm{div}\mathbf{H}=0$. Finally, from the induction equation in (1.5), we see that

$$\partial_t\mathbf{H}_\varepsilon=\nabla\times(\mathbf{u}_\varepsilon\times\mathbf{H}_\varepsilon)-\nabla\times(\nu_\varepsilon\nabla\times\mathbf{H}_i)$$

is bounded in $L^{8/7}([0,T];H^{-1}(\mathbb{T}))$, since $\mathbf{u}_\varepsilon$ is bounded in $L^2([0,T];L^4(\mathbb{T}))$, which implies that $\mathbf{u}_\varepsilon\times\mathbf{H}_\varepsilon$ and $\nu_\varepsilon\nabla\times\mathbf{H}_i$ are bounded in $L^{8/7}([0,T];L^2(\mathbb{T}))$. Then the Aubin–Lions compactness lemma (see [24]) implies that

$$\mathbf{H}_\varepsilon\to\mathbf{H}\quad\text{strongly in}\quad L^{8/7}([0,T];L^2(\mathbb{T})).$$

Moreover, this, combined with the uniform bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T];L^2(\mathbb{T}))$, implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^2([0,T];L^2(\mathbb{T}))$. Therefore, by a standard argument, we deduce that the limits $\mathbf{u}$ and $\mathbf{H}$ satisfy the induction equation in the sense of distributions, and also the nonlinear term $(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon$ in the second equation of (1.5) converges to $(\nabla\times\mathbf{H})\times\mathbf{H}$ in the sense of distributions.

Next, we claim that ρ_ε converges to 1 in $C([0,T];L^\gamma(\mathbb{T}))$. Indeed, in view of (3.1) and (3.2), we have

$$\begin{aligned}\sup_{t\ge0}\int_{\mathbb{T}}|\rho_\varepsilon-1|^\gamma dx&\le\delta^\gamma(2\pi)^N+C\sup_{t\ge0}\left(\int_{\mathbb{T}}\chi_{\{|\rho_\varepsilon-1|\ge\delta\}}|\rho_\varepsilon-\overline{\rho_\varepsilon}|^\gamma dx\right)+C|\overline{\rho_\varepsilon}-1|^\gamma\\ &\le(2\pi)^N\delta^\gamma+\frac{C\varepsilon^2}{\eta}+C|\overline{\rho_\varepsilon}-1|^\gamma,\end{aligned}$$

and we conclude the claim upon first letting ε go to 0 and then δ go to 0.

Now, we show from the previous bounds that $\mathrm{div}\mathbf{u}_\varepsilon$ converges weakly to 0 in $L^2([0,T];L^2(\mathbb{T}))$ and that $P\mathbf{u}_\varepsilon$ converges to $\mathbf{u}=P\mathbf{u}$ strongly in $L^2([0,T];L^2(\mathbb{T}))$, and thus by Sobolev imbedding in $L^2([0,T];L^q)$ for all $2\le q<\frac{2N}{N-2}$. These facts imply that $Q\mathbf{u}_\varepsilon$ converges weakly to 0 in $L^2([0,T];H^1(\mathbb{T}))$. Indeed, since ρ_ε converges to 1 in $C((0,\infty);L^\gamma(\mathbb{T}))$ and $\gamma>\frac{N}{2}$, we deduce from (1.5) that $\mathrm{div}\mathbf{u}_\varepsilon$ converges weakly to 0 in $L^2([0,T];L^2(\mathbb{T}))$. The second part is proved by observing first that we project (1.5) onto divergence-free vector fields:

$$\partial_tP(\rho_\varepsilon\mathbf{u}_\varepsilon)+P[\mathrm{div}(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)]-\mu_\varepsilon\Delta P\mathbf{u}_\varepsilon=P((\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon).\tag{3.3}$$

Noticing the fact that the operator P is bounded in all Sobolev space $W^{s,p}$ for all $s \in [0,\infty)$ and $1 < p < \infty$ and the preceding bounds, (3.3) yields a bound on $\partial_t P(\rho_\varepsilon \mathbf{u}_\varepsilon)$ in $L^1([0,T];H^{-1}(\mathbb{T})) + L^2([0,T];L^1(\mathbb{T})) + L^2([0,T];H^{-1}(\mathbb{T}))$ and hence in $L^1([0,T];H^{-1}(\mathbb{T}))$. In addition, $P(\rho_\varepsilon \mathbf{u}_\varepsilon)$ is bounded in $L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T})) \cap L^2([0,T];L^r(\mathbb{T}))$ with

$$\frac{1}{r} = \frac{1}{\gamma} + \frac{N-2}{2N}.$$

Next, we will need the following compactness lemma (cf. Lemma 5.1 in [24]).

Lemma 3.1. *Let g_n, h_n converge weakly to g, h, respectively, in $L^{p_1}(0,T;L^{p_2})$, $L^{q_1}(0,T;L^{q_2})$, where $1 \le p_1,\ p_2 \le \infty$:*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Assume, in addition, that

$$\frac{\partial g_n}{\partial t} \quad \text{is bounded in} \quad L^1(0,T;W^{-m,1}) \quad \text{for some} \quad m \ge 0 \quad \text{independent of } n$$

and

$$\|h_n - h_n(\cdot + \xi, t)\|_{L^{q_1}(0,T;L^{q_2})} \to 0 \quad \text{as} \quad |\xi| \to 0 \quad \text{uniformly in } n.$$

Then $g_n h_n$ converges to gh in the sense of distributions in $\Omega \times (0,T)$.

Applying this lemma with the previous bounds, we deduce that $P(\rho_\varepsilon \mathbf{u}_\varepsilon) \cdot P\mathbf{u}_\varepsilon$ converges in the sense of distributions to $|\mathbf{u}|^2$. We then easily conclude that $P\mathbf{u}_\varepsilon$ converges in $L^2([0,T];L^2(\mathbb{T}))$ to $\mathbf{u}$ upon using the weak convergence of $P\mathbf{u}_\varepsilon$ to $\mathbf{u}$ in $L^2([0,T];L^2(\mathbb{T}))$ and remarking that we have

$$\left| \int_0^T \int_{\mathbb{T}} \left(|P\mathbf{u}_\varepsilon|^2 - P(\rho_\varepsilon \mathbf{u}_\varepsilon) \cdot P\mathbf{u}_\varepsilon \right) dxdt \right| \le C \|\rho_\varepsilon - 1\|_{C([0,T];L^\gamma)} \|\mathbf{u}_\varepsilon\|^2_{L^2([0,T];L^s)},$$

with $s = \frac{2\gamma}{\gamma-1} < \frac{2N}{N-2}$ since $\gamma > \frac{N}{2}$.

We conclude this first step by showing the following bounds valid for all $R \in (1,\infty)$:

$$\text{(3.4)} \qquad \begin{cases} \|\varphi_\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{T}))} \le C & \text{if} \quad \gamma \ge 2, \\ \|\varphi_\varepsilon \chi_{\{\rho_\varepsilon < R\}}\|_{L^\infty([0,T];L^2(\mathbb{T}))} \le C & \text{if} \quad \gamma < 2, \\ \|\varphi_\varepsilon \chi_{\{\rho_\varepsilon \ge R\}}\|_{L^\infty([0,T];L^\gamma(\mathbb{T}))} \le C\varepsilon^{\frac{2}{\gamma}-1} & \text{if} \quad \gamma < 2, \end{cases}$$

where we denote the density fluctuation by $\varphi_\varepsilon = \frac{1}{\varepsilon}(\rho_\varepsilon - \overline{\rho_\varepsilon})$. These bounds are deduced immediately from the following straightforward inequalities: for some $\nu > 0$ and for all $x \ge 0$,

$$\text{(3.5)} \qquad \begin{cases} x^\gamma - 1 - \gamma(x-1) \ge \nu|x-1|^2 & \text{if} \quad \gamma \ge 2, \\ x^\gamma - 1 - \gamma(x-1) \ge \nu|x-1|^2 & \text{if} \quad \gamma < 2 \quad \text{and} \quad x \le R, \\ x^\gamma - 1 - \gamma(x-1) \ge \nu|x-1|^\gamma & \text{if} \quad \gamma < 2 \quad \text{and} \quad x \ge R. \end{cases}$$

3.2. The weak convergence of $Q\mathbf{u}$. The proof of the weak convergence of $Q\mathbf{u}$ is similar to that in Lions and Masmoudi [25]; thus we describe only briefly the main idea from [25]. We provide here first a formal proof of the passage to the limit, then the main difficulty, and finally the strategy of proof used in order to circumvent that difficulty.

We thus begin by an informal proof. It is not difficult to check that the main difficulty with the passage to the limit lies with the term $\operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$ and more precisely with the term $\operatorname{div}(\rho_\varepsilon Q(\mathbf{u}_\varepsilon) \otimes Q\mathbf{u}_\varepsilon)$ because of the strong convergence of $P\mathbf{u}_\varepsilon$. Formally, this term should not create an obstruction since in view of the continuity equation in (1.1), we can rewrite the term $\partial_t(\rho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$ as $\rho_\varepsilon \partial_t \mathbf{u}_\varepsilon + \rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon$, which corresponds to the term $\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}$ in incompressible MHD equations (1.6). Next, the dangerous term $[(Q\mathbf{u}_\varepsilon) \cdot \nabla]Q\mathbf{u}_\varepsilon$ can be incorporated in the pressure p at the limit since $Q\mathbf{u}_\varepsilon = \nabla\psi_\varepsilon$ for some ψ_ε, and then

$$[(Q\mathbf{u}_\varepsilon) \cdot \nabla]Q\mathbf{u}_\varepsilon = \nabla \left| \frac{1}{2} \nabla \psi_\varepsilon \right|^2 .$$

Next, we need to write down rigorously the proof of the convergence. First, we introduce the group $\{\mathcal{L}(t), t \in R\}$ defined by e^{tL}, where L is the operator defined on $\mathcal{D}'_0 \times (\mathcal{D}')^N$, where $\mathcal{D}'_0 = \{\phi \in \mathcal{D}', \int \phi = 0\}$, by

$$L\begin{pmatrix} \phi \\ v \end{pmatrix} = -\begin{pmatrix} \operatorname{div} v \\ b\nabla\phi \end{pmatrix} \quad \text{for} \quad b > 0. \tag{3.6}$$

We remark that e^{tL} is an isometry on each $H^s \times (H^s)^N$ for all $s \in R$ and for all t, endowed with the norm $\|(\phi, v)\| = (\|\phi\|^2_{H^s} + \frac{1}{b}\|v\|^2_{H^s})^{1/2}$. For details, we refer the reader to [25]. For convenience, in what follows, we will denote by $\mathcal{L}_1$ ($\mathcal{L}_2$) the first (the second) component of the operator $\mathcal{L}$, respectively.

We next claim that $\mathcal{L}(-\frac{t}{\varepsilon})\binom{\varphi_\varepsilon}{Q(\rho_\varepsilon \mathbf{u}_\varepsilon)}$ is relatively compact in $L^2([0,T]; H^{-n})$ for some $n \in (0,1)$. To this end, we need to prove first that $\binom{\varphi_\varepsilon}{Q(\rho_\varepsilon \mathbf{u}_\varepsilon)}$ is bounded in $L^2([0,T]; H^{-s})$ for some $s \in (0,1)$ and that $\partial_t\{\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\binom{\varphi_\varepsilon}{Q(\rho_\varepsilon \mathbf{u}_\varepsilon)}\}$ is bounded in $L^2([0,T]; H^{-r})$ for some $r > 0$ large enough. Our claim then follows from the Aubin–Lions compactness lemma by choosing n in $(s, 1)$.

Since we know that φ_ε is bounded in $L^\infty([0,T]; L^p)$, where $p = \min(2, \gamma)$, by Sobolev imbedding theorems, we know that φ_ε is bounded in $L^2([0,T]; H^{-s})$ for some $s \in (0,1]$. And we also deduce from the previous subsection that $\rho_\varepsilon \mathbf{u}_\varepsilon$ and thus $Q(\rho_\varepsilon \mathbf{u}_\varepsilon)$ are bounded in $L^2([0,T]; L^q)$ with

$$\frac{1}{q} = \frac{1}{\gamma} + \frac{N-2}{2N}.$$

Therefore, $\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\binom{\varphi_\varepsilon}{Q(\rho_\varepsilon \mathbf{u}_\varepsilon)}$ is bounded in $L^2([0,T]; H^{-s})$ for some $s \in (0,1)$.

In order to get the uniform bound on $\partial_t\{\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\binom{\varphi_\varepsilon}{Q(\rho_\varepsilon \mathbf{u}_\varepsilon)}\}$, we project the second equation of (1.5) into the space of gradient vector fields and find

$$\begin{aligned} &\partial_t Q(\rho_\varepsilon \mathbf{u}_\varepsilon) + Q[\operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] - (\mu_\varepsilon + \lambda_\varepsilon)\nabla \operatorname{div} \mathbf{u}_\varepsilon \\ &\quad + \frac{a}{\varepsilon^2}\nabla\left(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon(\overline{\rho_\varepsilon})^{\gamma-1} + (\gamma-1)(\overline{\rho_\varepsilon})^\gamma\right) + \frac{a\gamma(\overline{\rho_\varepsilon})^{\gamma-1}}{\varepsilon^2}\nabla(\rho_\varepsilon - \overline{\rho_\varepsilon}) \\ &= Q[(\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon]. \end{aligned} \tag{3.7}$$

Hence, we can write the first equation (1.5) and (3.7) as

$$\begin{aligned}&\varepsilon\frac{\partial\varphi_\varepsilon}{\partial t}+\mathrm{div}Q(\rho_\varepsilon\mathbf{u}_\varepsilon)=0,\\&\varepsilon\frac{\partial Q(\rho_\varepsilon\mathbf{u}_\varepsilon)}{\partial t}+b\nabla\varphi_\varepsilon=\varepsilon F_\varepsilon,\end{aligned}$$

where $b=a\gamma(\overline{\rho_\varepsilon})^{\gamma-1}$, and

$$\begin{aligned}F_\varepsilon=&(\mu_\varepsilon+\lambda_\varepsilon)\nabla\mathrm{div}\mathbf{u}_\varepsilon-Q[\mathrm{div}(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)]\\&-a\nabla\left(\frac{1}{\varepsilon^2}(\rho_\varepsilon^\gamma-\gamma\rho_\varepsilon(\bar{(}\rho_\varepsilon))^{\gamma-1}+(\gamma-1)(\overline{\rho_\varepsilon})^\gamma)+Q[(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon]\right).\end{aligned}$$

Since b goes to $a\gamma$ as ε goes to 0, we will ignore the dependence of b on ε hereafter. Now, we set

$$\psi_\varepsilon(t)=\mathcal{L}_1\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix}\varphi_\varepsilon\\Q(\rho_\varepsilon\mathbf{u}_\varepsilon)\end{pmatrix},\quad m_\varepsilon(t)=\mathcal{L}_2\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix}\varphi_\varepsilon\\Q(\rho_\varepsilon\mathbf{u}_\varepsilon)\end{pmatrix};$$

then we have

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}\psi_\varepsilon\\m_\varepsilon\end{pmatrix}&=\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\left\{\frac{\partial}{\partial t}\begin{pmatrix}\varphi_\varepsilon\\Q(\rho_\varepsilon\mathbf{u}_\varepsilon)\end{pmatrix}+\frac{1}{\varepsilon}\begin{pmatrix}\mathrm{div}Q(\rho_\varepsilon\mathbf{u}_\varepsilon)\\b\nabla\varphi_\varepsilon\end{pmatrix}\right\}\\&=\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix}0\\F_\varepsilon\end{pmatrix},\end{aligned}$$

where F_ε is bounded in $L^2([0,T];H^{-1}(\mathbb{T}))+L^2([0,T];W^{-1-\delta,1}(\mathbb{T}))$ for all $\delta>0$ and hence is bounded in $L^2([0,T];H^{-r}(\mathbb{T}))$ for all $r>\frac{N}{2}+1$. Thus $\frac{\partial}{\partial t}\binom{\psi_\varepsilon}{m_\varepsilon}$ is bounded in $L^2([0,T];H^{-r}(\mathbb{T}))$.

We deduce from the compactness of $(\psi_\varepsilon,m_\varepsilon)$ that we may assume without loss of generality that $(\psi_\varepsilon,m_\varepsilon)$ converges in $L^2([0,T];H^{-n})$ to some (ψ,m). Since $Pm_\varepsilon=0$, we also have $Pm=0$. Similarly, $\int\psi=0$. Hence, we have

$$\text{(3.8)}\quad\begin{pmatrix}\varphi_\varepsilon\\Q(\rho_\varepsilon\mathbf{u}_\varepsilon)\end{pmatrix}=\mathcal{L}\left(\frac{t}{\varepsilon}\right)\begin{pmatrix}\varphi\\m\end{pmatrix}+r_\varepsilon,\quad r_\varepsilon\to0\quad\text{in}\quad L^2([0,T];H^{-n})\quad\text{as}\quad\varepsilon\to0.$$

Finally, following the argument of Step 4 in section 3 in [25], one can show that φ, $m\in L^2([0,T];L^2(\mathbb{T}))$ and $\mathrm{div}(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)-\mathrm{div}(v_\varepsilon\otimes v_\varepsilon)$ converges to $\mathrm{div}(\mathbf{u}\otimes\mathbf{u})$ in the sense of distributions, where $v_\varepsilon=\mathcal{L}_2(\frac{t}{\varepsilon})\binom{\varphi}{m}$. Moreover, following the argument of Step 5 in section 3 in [25], we can show that $\mathrm{div}(v_\varepsilon\otimes v_\varepsilon)$ converges to a distribution which is a gradient. Note that the magnetic field does not affect the argument of convergence of $\mathrm{div}(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)-\mathrm{div}(v_\varepsilon\otimes v_\varepsilon)$ and $\mathrm{div}(v_\varepsilon\otimes v_\varepsilon)$ because the magnetic field $\mathbf{H}$ does not affect the integrability of F_ε based on our estimates; thus we state only those convergence results without proof. We refer the readers to [25] for details.

This finishes the proof of Theorem 2.1. ☐

4. The whole space case. In this section, we prove Theorem 2.2. The idea is taken from [5]. Before we start, we introduce homogeneous Sobolev spaces for $1<p<\infty$ and $s\in R$ defined as usual by

$$\dot{W}^{s,p}(\mathbb{R}^N)=(-\Delta)^{-s/2}L^p(\mathbb{R}^N)\quad\text{and}\quad\dot{H}^s(\mathbb{R}^N)=\dot{W}^{s,2}(\mathbb{R}^N),$$

where Δ is the Laplace operator.

Let us denote by $\zeta \in C_0^\infty(\mathbb{R}^N)$ a smoothing kernel such that $\zeta \geq 0$, $\int_{\mathbb{R}^N} \zeta dx = 1$, and define $\zeta_\alpha(x) = \alpha^{-N}\zeta(x/\alpha)$. The following estimate will be useful in this section (cf. [5]):

$$\|f - f * \zeta_\alpha\|_{L^q} \leq C\alpha^{1-\sigma}\|\nabla f\|_{L^2} \quad \text{for all} \quad f \in \dot{H}^1, \tag{4.1}$$

where

$$q \in \left[2, \frac{2N}{N-2}\right) \quad \text{and} \quad \sigma = N\left(\frac{1}{2} - \frac{1}{q}\right),$$

and for $1 < p_2 < p_1 < \infty$, $s \geq 0$, and $\alpha \in (0,1)$, we have

$$\|g * \zeta_\alpha\|_{L^{p_1}(\mathbb{R}^N)} \leq C\alpha^{-s-N(1/p_2-1/p_1)}\|g\|_{W^{-s,p_2}(\mathbb{R}^N)}. \tag{4.2}$$

4.1. A priori estimates and consequences. Most of the arguments developed in the periodic case can be adapted to the whole space case. First, we obtain bounds on $D\mathbf{u}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{R}^N))$, on $\nabla \times \mathbf{H}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{R}^N))$, and on $\rho_\varepsilon|\mathbf{u}_\varepsilon|^2$ and $\frac{1}{\varepsilon^2}(\rho_\varepsilon^\gamma + (\gamma-1) - \gamma\rho_\varepsilon)$ in $L^\infty([0,T];L^1(\mathbb{R}^N))$. The bound on $\mathbf{u}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{R}^N))$ follows from (3.5) and the following observation:

$$\int_{R^N}\left(\frac{1}{\varepsilon^2}|\rho_\varepsilon - 1|^2\chi_{\{|\rho_\varepsilon-1|\leq 1/2\}} + \frac{1}{\varepsilon^2}|\rho_\varepsilon - 1|^\gamma\chi_{\{|\rho_\varepsilon-1|\geq 1/2\}}\right) \leq C, \tag{4.3}$$

and thus, in particular,

$$\begin{aligned}
\int_{\mathbb{R}^N} |\mathbf{u}_\varepsilon|^2 dx &\leq C + \int_{\mathbb{R}^N} |\mathbf{u}_\varepsilon|^2\chi_{\{\rho_\varepsilon \leq 1/2\}}dx \\
&\leq C + \left(\int_{\mathbb{R}^N} \chi_{\{\rho_\varepsilon\leq 1/2\}}dx\right)^{1/\gamma}\left(\int_{\mathbb{R}^N}|\mathbf{u}_\varepsilon|^{2\gamma'}dx\right)^{1/\gamma'} \\
&\leq C\left(1 + (\text{meas}(|\rho_\varepsilon - 1| \geq 1/2))^{1/\gamma}\|\mathbf{u}_\varepsilon\|_{L^2}^{2\theta}\|D\mathbf{u}_\varepsilon\|_{L^2}^{2(1-\theta)}\right) \\
&\leq C\left(1 + \varepsilon^{2/\gamma}\|\mathbf{u}_\varepsilon\|_{L^2}^{2\theta}\|D\mathbf{u}_\varepsilon\|_{L^2}^{2(1-\theta)}\right),
\end{aligned}$$

where

$$\frac{\theta}{2} + (1-\theta)\frac{N-2}{2N} = \frac{1}{2\gamma'}.$$

We then complete the proof of our claim using the bound on $D\mathbf{u}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{R}^N))$ and the classical Young inequality. Moreover, if we define the density fluctuation as

$$\varphi_\varepsilon = \frac{\rho_\varepsilon - 1}{\varepsilon},$$

then it is bounded uniformly in ε in $L^\infty([0,T];L_2^\kappa)$ with $\kappa = \min\{2,\gamma\}$. Furthermore, if we write

$$\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon^1 + \mathbf{u}_\varepsilon^2,$$

where

$$\mathbf{u}_\varepsilon^1 = \mathbf{u}_\varepsilon\chi_{\{|\rho_\varepsilon-1|\leq 1/2\}} \quad \text{and} \quad \mathbf{u}_\varepsilon^2 = \mathbf{u}_\varepsilon\chi_{\{|\rho_\varepsilon-1|>1/2\}},$$

then we have

$$\sup_{t\geq 0}\int_{\mathbb{R}^N}|\mathbf{u}_\varepsilon^1|^2dx \leq 2\sup_{t\geq 0}\int_{\mathbb{R}^N}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2dx \leq C,$$

and for $p<\kappa$ when $N=2$, $p=2\kappa/3$ if $N=3$,

$$\begin{aligned}\int_{\mathbb{R}^N}|\mathbf{u}_\varepsilon^2|^2dx &\leq C\int_{\mathbb{R}^N}|\rho_\varepsilon-1|^p\chi_{\{|\rho_\varepsilon-1|>1/2\}}|\mathbf{u}_\varepsilon|^2dx\\ &\leq C\|(\rho_\varepsilon-1)\chi_{\{|\rho_\varepsilon-1|>1/2\}}\|^p_{L^\infty([0,T];L^\kappa(\mathbb{R}^N))}\|\mathbf{u}_\varepsilon\|^2_{L^{2\kappa/(\kappa-p)}}\\ &\leq C\varepsilon^{2p/\kappa}\|\mathbf{u}_\varepsilon\|^{2-pN/\kappa}_{L^2(\mathbb{R}^N)}\|\nabla\mathbf{u}_\varepsilon\|^{pN/\kappa}_{L^2(\mathbb{R}^N)};\end{aligned}$$

hence, by Young's inequality, $\mathbf{u}_\varepsilon^1$ is bounded in $L^\infty([0,T];L^2(\mathbb{R}^N))$ and $\mathbf{u}_\varepsilon^2\varepsilon^{-\beta}$ is bounded in $L^2([0,T];L^2(\mathbb{R}^N))$, where $\beta\in(0,1)$ if $N=2$ and $\beta=2/3$ if $N=3$.

Recalling that $\gamma>N/2$, we deduce that $\mathbf{u}_\varepsilon$ is bounded in

$$L^2([0,T];L^4(\mathbb{R}^N)\cap L^{2\gamma/(\gamma-1)}(\mathbb{R}^N)).$$

Hence, we have

$$\|\varphi_\varepsilon\mathbf{u}_\varepsilon\|_{L^2([0,T];L^{4/3}(\mathbb{R}^N)+L^{2\kappa/(\kappa+1)}(\mathbb{R}^N))}\leq C.$$

Therefore, using Sobolev's imbedding, we deduce that

$$\|\varphi_\varepsilon\mathbf{u}_\varepsilon\|_{L^2([0,T];H^{-1}(\mathbb{R}^N))}\leq C.$$

Finally, we already know that φ_ε^0 is bounded in $L_2^\kappa(\mathbb{R}^N)$, hence in $H^{-1}(\mathbb{R}^N)$, since $\gamma>N/2$. On the other hand, m_ε^0 can be rewritten as

$$m_\varepsilon^0=\frac{m_\varepsilon^0}{\sqrt{\rho_\varepsilon^0}}\sqrt{\rho_\varepsilon^0}\chi_{\{|\rho_\varepsilon^0-1|\leq 1/2\}}+\frac{m_\varepsilon^0}{\sqrt{\rho_\varepsilon^0}}\frac{\sqrt{\rho_\varepsilon^0}}{\sqrt{|\rho_\varepsilon^0-1|}}\sqrt{|\rho_\varepsilon^0-1|}\chi_{\{|\rho_\varepsilon^0-1|>1/2\}}.$$

This implies that m_ε^0 is bounded in $L^2(\mathbb{R}^N)+L^{2\kappa/(\kappa+1)}(\mathbb{R}^N)$ and hence in $H^{-1}(\mathbb{R}^N)$. Therefore, $\binom{\varphi_\varepsilon^0}{m_\varepsilon^0}$ is bounded in $H^{-1}(\mathbb{R}^N)$ uniformly in ε.

4.2. Strong convergence of $Q\mathbf{u}_\varepsilon$ to 0. We now prove that the gradient part of the velocity $Q\mathbf{u}_\varepsilon$ converges strongly to 0. More precisely, we claim that $Q\mathbf{u}_\varepsilon$ converges strongly to 0 in $L^2([0,T];L^p(\mathbb{R}^N))$ for all $p\in(2,\frac{2N}{N-2})$. Indeed, let us first observe that the compressible MHD equations can be rewritten in terms of the density fluctuation φ_ε, the momentum $m_\varepsilon=\rho_\varepsilon\mathbf{u}_\varepsilon$, and $\phi_\varepsilon=\binom{\varphi_\varepsilon}{m_\varepsilon}$ as follows:

$$\partial_t\phi_\varepsilon+\frac{L\phi_\varepsilon}{\varepsilon}=F_\varepsilon^1+F_\varepsilon^2,$$

where the wave operator L is defined on $(\mathcal{D}'(\mathbb{R}^N))^{N+1}$ with values in $(\mathcal{D}'(\mathbb{R}^N))^{N+1}$ by

$$L\phi=\begin{pmatrix}\operatorname{div}m\\ \nabla\psi\end{pmatrix},\quad\text{with}\quad\phi=\begin{pmatrix}\psi\\ m\end{pmatrix},$$

and

$$F_\varepsilon^1=\begin{pmatrix}0\\ \mu_\varepsilon\Delta\mathbf{u}_\varepsilon^1+\lambda_\varepsilon\nabla\mathrm{div}\mathbf{u}_\varepsilon^1-\mathrm{div}(m_\varepsilon\otimes\mathbf{u}_\varepsilon)-\frac{a}{\varepsilon^2}\nabla(\rho_\varepsilon^\gamma-1-\gamma(\rho_\varepsilon-1))+(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon\end{pmatrix},$$

$$F_\varepsilon^2=\begin{pmatrix}0\\ \mu_\varepsilon\Delta\mathbf{u}_\varepsilon^2+\lambda_\varepsilon\nabla\mathrm{div}\mathbf{u}_\varepsilon^2\end{pmatrix}.$$

Using *Duhamel*'s formula, we deduce that

$$Q\phi_\varepsilon(t) = \mathcal{L}\left(\frac{t}{\varepsilon}\right) Q\phi_\varepsilon^0 + \int_0^t \mathcal{L}\left(\frac{t-s}{\varepsilon}\right)(QF_\varepsilon^1(s) + QF_\varepsilon^2(s))ds.$$

Here we used the fact that Q and $\mathcal{L}$ commute, since Q and L do.

At this stage, the following Strichartz estimates from [5] are useful.

LEMMA 4.1. *For all $s \geq 0$, we have*

$$\left\|\mathcal{L}\left(\frac{t}{\varepsilon}\right) Q\psi_0\right\|_{L^q((0,\infty);W^{-s-\sigma,p}(\mathbb{R}^N))} \leq C\varepsilon^{1/q}\|\psi_0\|_{H^{-s}(\mathbb{R}^N)}, \tag{4.4}$$

$$\left\|\int_0^t \mathcal{L}\left(\frac{t-s}{\varepsilon}\right) Q\psi(s)ds\right\|_{L^q([0,T];W^{-s-\sigma,p}(\mathbb{R}^N))} \leq C(1+T)\varepsilon^{1/q}\|\psi\|_{L^q([0,T];H^{-s}(\mathbb{R}^N))} \tag{4.5}$$

for all $(p,q) \in (2,\infty)\times(2,\infty)$ and $\sigma \in (0,\infty)$ such that

$$\frac{2}{q} = (N-1)\left(\frac{1}{2} - \frac{1}{p}\right) \quad \text{and} \quad \sigma q = \frac{N+1}{N-1}. \tag{4.6}$$

Now, we choose $p \in (2, \frac{2N}{N-2})$, $q \in (2,\infty)$, and $\sigma \in (0,\infty)$ given by (4.6). One can deduce that

$$|Q\mathbf{u}_\varepsilon| \leq |Q\mathbf{u}_\varepsilon - Q\mathbf{u}_\varepsilon * \zeta_\alpha| + \varepsilon|Q(\mathbf{u}_\varepsilon\varphi_\varepsilon) * \zeta_\alpha| + |Qm_\varepsilon * \zeta_\alpha|.$$

Hence,

$$\begin{aligned}\|Q\mathbf{u}_\varepsilon\|_{L^2([0,T];L^p(\mathbb{R}^N))} \leq{}& C\alpha^{1-N(1/2-1/p)}\|\nabla\mathbf{u}_\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^N))}\\ &+ \varepsilon\alpha^{-1-N(1/2-1/p)}\|\varphi_\varepsilon\mathbf{u}_\varepsilon\|_{L^2([0,T];H^{-1}(\mathbb{R}^N))}\\ &+ \|Qm_\varepsilon * \zeta_\alpha\|_{L^2([0,T];L^p(\mathbb{R}^N))}.\end{aligned}$$

We know that F_ε^1 is bounded in $L^\infty([0,T];H^{-s_0})$ for all $s_0 > N/2+1$ from the estimates in the previous subsection. On the other hand, we deduce from the uniform bound on $\mathbf{u}_\varepsilon^2\varepsilon^{-\beta}$ in $L^2([0,T];L^2)$ that $\varepsilon^{-\beta}F_\varepsilon^2$ is bounded in $L^2([0,T];H^{-2}(\mathbb{R}^N))$. Then, using Lemma 4.1, we obtain, for all $\eta > 0$ small enough,

$$\begin{aligned}&\|Qm_\varepsilon * \zeta_\alpha\|_{L^2([0,T];L^p(\mathbb{R}^N))}\\ &\leq C_T\alpha^{-1-\sigma}\left\|\mathcal{L}\left(\frac{t}{\varepsilon}\right)\psi_\varepsilon^0\right\|_{L^q([0,T];W^{-1-\sigma,p}(\mathbb{R}^N))}\\ &\qquad + C_T\alpha^{-N/2-1-\sigma-\eta}\left\|\int_0^T ds\mathcal{L}\left(\frac{t-s}{\varepsilon}\right)QF_\varepsilon^1(s)\right\|_{L^q([0,T];W^{-\eta-N/2-1-\sigma,p}(\mathbb{R}^N))}\\ &\qquad + C\alpha^{-2-N(1/2-1/p)}\left\|\int_0^t ds\mathcal{L}\left(\frac{t-s}{\varepsilon}\right)QF_\varepsilon^2(s)\right\|_{L^2([0,T];H^{-2})}\\ &\leq C_T\alpha^{-1-\sigma}\varepsilon^{1/q}\|\psi_\varepsilon^0\|_{H^{-1}} + C_T\alpha^{-N/2-1-\sigma-\eta}\varepsilon^{1/q}\|F_\varepsilon^1\|_{L^\infty([0,T];H^{-\eta-N/2-1})}\\ &\qquad + C\alpha^{-2-N(1/2-1/p)}\varepsilon^\beta\|\varepsilon^{-\beta}F_\varepsilon^2\|_{L^2([0,T];H^{-2})}.\end{aligned}$$

Next, fixing $\alpha > 0$ and letting ε go to zero, we obtain

$$\limsup_{\varepsilon\to 0}\|Q\mathbf{u}_\varepsilon\|_{L^2([0,T];L^p(\mathbb{R}^N))} \leq C\alpha^{1-N(1/2-1/p)},$$

where C is independent of ε and α. Noticing that $1 - N(1/2 - 1/p) > 0$, we finally get, by letting $\alpha \to 0$,

$$\limsup_{\varepsilon \to 0} \|Q\mathbf{u}_\varepsilon\|_{L^2([0,T];L^p(\mathbb{R}^N))} = 0.$$

This implies that $Q\mathbf{u}_\varepsilon$ strongly converges to 0 in $L^2([0,T]; L^p(\mathbb{R}^N))$ for all $2 < p < \frac{2N}{N-2}$.

4.3. Strong convergences of $P\mathbf{u}_\varepsilon$ and $\mathbf{H}_\varepsilon$. In the previous section, we proved the strong convergence of the gradient part of the velocity to 0. In order to complete the proof of Theorem 2.2, we are left to show the convergence of the incompressible part of the velocity, $P\mathbf{u}_\varepsilon$, the convergence of the density, and the convergence of the magnetic field. This can be done by using the classical compactness arguments in [24, 25], or equivalently by looking at the time-regularity properties of $P\mathbf{u}_\varepsilon$; see [5]. Indeed, following the argument in the periodic case step by step, we obtain the strong convergence of ρ_ε to 1 in $C([0,T]; L^\gamma_{loc}(\mathbb{R}^N))$ and the weak convergence of $P\mathbf{u}_\varepsilon$ to $\mathbf{u}$ in $L^2([0,T]; H^1(\mathbb{R}^N))$. Moreover, we also can show that $P\mathbf{u}_\varepsilon$ converges to $\mathbf{u}$ in $L^2([0,T]; L^2(B_R))$ for all $R \in (0,\infty)$. Here, we denote by B_R the open ball centered at 0 of radius R.

Finally, similarly as in the periodic case, the bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T]; L^2(\mathbb{R}^N))$, and the bound on $\nabla\mathbf{H}_\varepsilon$ in $L^2([0,T]; L^2(\mathbb{R}^N))$, combining Sobolev's inequality and the interpolation theorem, we know that $\mathbf{H}_\varepsilon$ is bounded in $L^{8/3}([0,T]; L^4(\mathbb{R}^N))$, and we can also assume that $\mathbf{H}_\varepsilon$ converges weakly to some $\mathbf{H}$ in $L^2([0,T]; H^1(\mathbb{R}^N))$ with $\mathrm{div}\mathbf{H} = 0$. Finally, from the induction equation in (1.5), we deduce that $\partial_t \mathbf{H}_\varepsilon$ is bounded in $L^{8/7}([0,T]; H^{-1}(\mathbb{R}^N))$, due to the fact that $\mathbf{u}_\varepsilon$ is bounded in $L^2([0,T]; L^4(\mathbb{R}^N))$. This property, combining the Aubin–Lions compactness lemma, implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^{8/7}([0,T]; L^2_{loc}(\mathbb{R}^N))$. Moreover, the uniform bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T]; L^2(\mathbb{R}^N))$ implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^2([0,T]; L^2_{loc}(\mathbb{R}^N))$. Therefore, by a standard argument, we deduce that the limits $\mathbf{u}$ and $\mathbf{H}$ satisfy the induction equation in (1.1) in the sense of distributions, and also the nonlinear term $(\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon$ in the second equation of (1.5) converges to $(\nabla \times \mathbf{H}) \times \mathbf{H}$ in the sense of distributions. For a detailed statement of the above argument, we refer the readers to the argument surrounding the convergence of the magnetic field in section 3.1.

The proof of Theorem 2.2 is complete. $\square$

5. The bounded domain case.

In this section, we will prove Theorem 2.3 by the spectral analysis of the semigroup [30] generated by the dissipative wave operator. Before we start, we introduce the eigenvalues $\{\lambda^2_{k,0}\}_{k\in\mathbb{N}}$ ($\lambda_{k,0} > 0$) and the eigenvectors $\{\Psi_{k,0}\}_{k\in\mathbb{N}}$ in $L^2(\Omega)$ with zero mean value of the Laplace operator satisfying the following homogeneous Neumann boundary conditions:

$$-\Delta\Psi_{k,0} = \lambda^2_{k,0}\Psi_{k,0} \quad \text{in } \Omega, \qquad \frac{\partial\Psi_{k,0}}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega.$$

Notice that, by the Gram–Schmidt orthogonalization method, it is possible to assume that $\{\Psi_{k,0}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and that up to a slight modification, if $\lambda_{k,0} = \lambda_{l,0}$ and $k \neq l$, then

$$\int_{\partial\Omega} \nabla\Psi_{k,0} \cdot \nabla\Psi_{l,0} ds = 0.$$

Next, we recall from the previous section that we can deduce similarly that

$$\sup_{t\geq 0}\|\rho_\varepsilon - 1\|_{L^\gamma(\Omega)} \leq C\varepsilon^{\kappa/\gamma} \quad \text{and} \quad \sup_{t\geq 0}\|\rho_\varepsilon - 1\|_{L^\kappa(\Omega)} \leq C\varepsilon,$$

where $\kappa = \min\{2,\gamma\}$. And similarly to the whole space case, we will split

$$\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon^1 + \mathbf{u}_\varepsilon^2, \quad \text{with} \quad \mathbf{u}_\varepsilon^1 = \mathbf{u}_\varepsilon \chi_{|\rho_\varepsilon - 1|\leq 1/2}, \quad \mathbf{u}_\varepsilon^2 = \mathbf{u}_\varepsilon \chi_{|\rho_\varepsilon - 1|>1/2},$$

which satisfy

$$\sup_{t\geq 0}\int_\Omega |\mathbf{u}_\varepsilon^1|^2 dx \leq 2\sup_{t\geq 0}\int_\Omega \rho_\varepsilon |\mathbf{u}_\varepsilon|^2 dx \leq C$$

and

$$\|\mathbf{u}_\varepsilon^2\|_{L^2(\Omega)}^2 \leq 2\int_\Omega |\rho_\varepsilon - 1||\mathbf{u}_\varepsilon|^2 dx \leq C\varepsilon\|\mathbf{u}_\varepsilon\|^2_{L^{2\kappa/(\kappa-1)}(\Omega)} \leq C\varepsilon|\nabla \mathbf{u}_\varepsilon|^2_{L^2(\Omega)}.$$

Therefore, $\mathbf{u}_\varepsilon^1$ is bounded in $L^\infty([0,T];L^2(\Omega))$, whereas $\mathbf{u}_\varepsilon^2\varepsilon^{-1/2}$ is bounded in $L^2(\Omega\times(0,T))$, and hence $\mathbf{u}_\varepsilon$ is bounded in $L^2(\Omega\times(0,T))$. Also, in this section, we denote the density fluctuation by

$$\varphi_\varepsilon = \frac{\rho_\varepsilon - 1}{\varepsilon}$$

and the momentum by $\mathbf{m}_\varepsilon = \rho_\varepsilon \mathbf{u}_\varepsilon$.

5.1. Strong convergence of $P\mathbf{u}_\varepsilon$ and $\mathbf{H}_\varepsilon$. Following the argument in the periodic case step by step, up to the extraction of a subsequence, we then obtain the strong convergence of ρ_ε to 1 in $C([0,T];L^\gamma(\Omega))$, the strong convergence of $P\mathbf{u}_\varepsilon$ to $\mathbf{u} = P\mathbf{u}$ in $L^2(\Omega\times(0,T))$, and the weak convergence of $Q\mathbf{u}_\varepsilon$ to 0 in $L^2([0,T];H^1(\Omega))$. Thus, the continuity equation in (1.5) holds in the sense of distributions.

Similarly to the periodic case, the bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T];L^2(\Omega))$, and the bound on $\nabla\mathbf{H}_\varepsilon$ in $L^2([0,T];L^2(\Omega))$, combining Sobolev's inequality and the interpolation theorem, we know that $\mathbf{H}_\varepsilon$ is bounded in $L^{8/3}([0,T];L^4(\Omega))$, and we can also assume that $\mathbf{H}_\varepsilon$ converges weakly to some $\mathbf{H}$ in $L^2([0,T];H^1(\Omega))$ with $\operatorname{div}\mathbf{H} = 0$. Also, from the induction equation in (1.5), we deduce that $\partial_t\mathbf{H}_\varepsilon$ is bounded in $L^{8/7}([0,T];H^{-1}(\Omega))$, due to the fact that $\mathbf{u}_\varepsilon$ is bounded in $L^2([0,T];L^4(\Omega))$. This property, combining the Aubin–Lions compactness lemma, implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^{8/7}([0,T];L^2(\Omega))$. Moreover, the uniform bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T];L^2(\Omega))$ implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^2([0,T];L^2(\Omega))$. Therefore, by a standard argument, we deduce that the limits $\mathbf{u}$ and $\mathbf{H}$ satisfy the induction equation in (1.1) in the sense of distributions, and also the nonlinear term $(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon$ in the second equation of (1.5) converges to $(\nabla\times\mathbf{H})\times\mathbf{H}$ in the sense of distributions. Therefore, in order to prove Theorem 2.3, it remains only to study the convergence of the gradient part of the velocity $Q\mathbf{u}_\varepsilon$.

5.2. The convergence of $Q\mathbf{u}_\varepsilon$. The argument for the convergence of $Q\mathbf{u}_\varepsilon$ in this subsection follows along the lines in [6], except for the argument for the magnetic field. For the readers' convenience and the completeness of the argument, we provide the details here. For this purpose, first, we discuss the spectral problem associated with the viscous wave operator L_ε in terms of eigenvalues and eigenvectors of the inviscid wave operator L, where the wave operators L and L_ε are defined on $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)^N$ by

$$L\begin{pmatrix}\psi\\ \mathbf{m}\end{pmatrix} = \begin{pmatrix}\mathrm{div}\mathbf{m}\\ \nabla\Psi\end{pmatrix}$$

and

$$L_\varepsilon\begin{pmatrix}\Psi\\ \mathbf{m}\end{pmatrix} = L\begin{pmatrix}\Psi\\ \mathbf{m}\end{pmatrix} + \varepsilon\begin{pmatrix}0\\ \mu_\varepsilon\Delta\mathbf{m} + \lambda_\varepsilon\nabla\mathrm{div}\mathbf{m}\end{pmatrix}.$$

The eigenvalues and eigenvectors of L read as follows:

$$\phi^{\pm}_{k,0} = \begin{pmatrix}\Psi_{k,0}\\ \mathbf{m_{k,0}} = \pm\frac{\nabla\boldsymbol{\Psi}_{\mathbf{k,0}}}{\mathrm{i}\lambda_{\mathbf{k,0}}}\end{pmatrix},$$

$$L\phi^{\pm}_{k,0} = \pm\mathrm{i}\lambda_{k,0}\phi^{\pm}_{k,0} \quad \text{in } \Omega, \qquad \mathbf{m}^{\pm}_{k,0}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

In the following steps, the following information on the approximating eigenvalues and eigenvectors for the operator L_ε is crucial.

LEMMA 5.1. *Let Ω be a C^2 bounded domain in R^N and let $k \geq 1$, $M \geq 0$. Then, there exist approximate eigenvalues $\mathrm{i}\lambda^{\pm}_{k,\varepsilon,M}$ and eigenvectors $\phi^{\pm}_{k,\varepsilon,M} = \binom{\Psi^{\pm}_{k,\varepsilon,M}}{\mathbf{m}^{\pm}_{k,\varepsilon,M}}$ of L_ε such that*

$$L_\varepsilon\phi^{\pm}_{k,\varepsilon,M} = \mathrm{i}\lambda^{\pm}_{k,\varepsilon,M}\phi^{\pm}_{k,\varepsilon,M} + R^{\pm}_{k,\varepsilon,M},$$

with

$$\mathrm{i}\lambda^{\pm}_{k,\varepsilon,M} = \pm\mathrm{i}\lambda_{k,0} + \mathrm{i}\lambda^{\pm}_{k,1}\sqrt{\varepsilon} + O(\varepsilon), \qquad \text{where } \mathcal{R}e(\mathrm{i}\lambda^{\pm}_{k,1}) \leq 0,$$

and for all $1 \leq p \leq \infty$ we have

$$\|R^{\pm}_{k,\varepsilon,M}\|_{L^p(\Omega)} \leq C_p(\sqrt{\varepsilon})^{M+1/p} \quad \text{and} \quad \|\phi^{\pm}_{k,\varepsilon,M} - \phi^{\pm}_{k,0}\|_{L^p(\Omega)} \leq C_p(\sqrt{\varepsilon})^{1/p}.$$

Proof. For the construction in detail, we refer the readers to [6]. ☐

Remark 5.1. Due to the construction in [6], indeed, we have

$$\mathrm{i}\lambda^{\pm}_{k,1} = -\frac{1\pm\mathrm{i}}{2}\sqrt{\frac{\mu_\varepsilon}{2\lambda^3_{k,0}}}\int_{\partial\Omega}|\nabla\Psi_{k,0}|^2\,ds.$$

Remark 5.2. We notice that the first order term $\mathrm{i}\lambda^{\pm}_{k,1}$ clearly yields an instantaneous damping of the acoustic waves, as soon as $\mathcal{R}e(\mathrm{i}\lambda^{\pm}_{k,1}) < 0$. For this reason, we define $I \subset \mathbb{N}$ to be the set of eigenvectors $\Psi_{k,0}$ of the Laplace operator such that $\mathcal{R}e(\mathrm{i}\lambda^{\pm}_{k,1}) < 0$ and $J = \mathbb{N} - I$. Observe that when $k \in J$, we have $\lambda^{\pm}_{k,1} = 0$. For those indices, $\mathbf{m}^{\pm}_{k,0}$ identically vanishes on $\partial\Omega$ and therefore satisfies not only $\mathbf{m}^{\pm}_{k,0}\cdot\mathbf{n} = \mathbf{0}$ but also $\mathbf{m}^{\pm}_{k,0} = 0$ on $\partial\Omega$; hence no significant boundary layer is created, and there is no enhanced dissipation of energy in these layers.

Now, we can express $Q\mathbf{u}_\varepsilon$ in the terms of the orthonormal basis $\{\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}}\}_{k\in\mathbf{N}}$ of $L^2(\Omega)$ as

$$Q\mathbf{u}_\varepsilon = \sum_{k\in\mathbf{N}}\left(Q\mathbf{u}_\varepsilon, \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}}\right)\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}},$$

where the notation $(\cdot,\cdot)$ stands for

$$(f(x), g(x)) = \int_\Omega f(x)\overline{g(x)}dx.$$

We can split $Q\mathbf{u}_\varepsilon$ into two parts $Q_1\mathbf{u}_\varepsilon$ and $Q_2\mathbf{u}_\varepsilon$, defined by

$$Q_1\mathbf{u}_\varepsilon = \sum_{k\in I}\left(Q\mathbf{u}_\varepsilon, \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}}\right)\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \quad \text{and} \quad Q_2\mathbf{u}_\varepsilon = \sum_{k\in J}\left(Q\mathbf{u}_\varepsilon, \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}}\right)\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}},$$

which, respectively, correspond to damped terms and nondamped terms. We will prove on the one hand that $Q_1\mathbf{u}_\varepsilon$ converges strongly to 0 in $L^2(\Omega\times(0,T))$, and on the other hand that $\mathrm{curldiv}(Q_2m_\varepsilon\otimes Q_2\mathbf{u}_\varepsilon)$ converges to 0 in the sense of distributions, if $J\neq\emptyset$, which is equivalent to saying that $\mathrm{div}(Q_2m_\varepsilon\otimes Q_2\mathbf{u}_\varepsilon)$ converges to a gradient in the sense of distributions.

Let us observe that in view of the bound on $\mathbf{u}_\varepsilon$ in $L^2([0,T];H_0^1(\Omega))$, the problem reduces to a finite number of terms. Indeed, we have

$$\sum_{k>M}\int_0^T\left|\left(Q_i\mathbf{u}_\varepsilon, \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}}\right)\right|^2 dt \le \frac{C}{\lambda_{M+1}^2}|\nabla\mathbf{u}_\varepsilon|^2_{L^2(\Omega\times(0,T))}, \quad i = 1 \text{ or } 2.$$

Hence, recalling that $\lambda_M\to\infty$ as $M\to\infty$, we have to prove only that $(Q_1\mathbf{u}_\varepsilon, \mathbf{m}^\pm_{k,0})$ converges strongly to 0 in $L^2(0,T)$ for any fixed k and study the interaction of a finite number of terms in $\mathrm{div}(Q_2\mathbf{u}_\varepsilon\otimes Q_2\mathbf{u}_\varepsilon)$. On the other hand, we notice that

$$Q\mathbf{u}_\varepsilon = Q\mathbf{m}_\varepsilon - \varepsilon Q(\varphi_\varepsilon\mathbf{u}_\varepsilon)$$

and

$$\begin{aligned}\varepsilon|(Q(\varphi_\varepsilon\mathbf{u}_\varepsilon), \nabla\Psi_{k,0})| &= \varepsilon\left|\int_\Omega \varphi_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla\Psi_{k,0}dx\right| \\ &\le \varepsilon\|\varphi_\varepsilon\|_{L^\gamma(\Omega)}\|\mathbf{u}_\varepsilon\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\|\nabla\Psi_{k,0}\|_{L^\infty(\Omega)},\end{aligned}$$

which goes to 0 in $L^2(0,T)$ since $\gamma>N/2$. Hence, we are led to study $(Q\mathbf{m}_\varepsilon, \mathbf{m}^\pm_{k,0})$.

Denoting

$$\beta^\pm_{k,\varepsilon} = (\phi_\varepsilon(t), \phi^\pm_{k,0}), \quad \text{with} \quad \phi_\varepsilon(t) = \begin{pmatrix}\varphi_\varepsilon\\ \mathbf{m}_\varepsilon\end{pmatrix},$$

we observe that

$$2(Q\mathbf{m}_\varepsilon, \mathbf{m}^\pm_{k,0}) = \beta^\pm_{k,\varepsilon} - \beta^\mp_{k,\varepsilon},$$

so that it suffices to consider the convergence properties of $\beta^\pm_{k,\varepsilon}$ in $L^2(0,T)$. Also we see from Lemma 5.1 with $M=2$ that

$$|(\phi_\varepsilon(t), \phi^\pm_{k,0} - \phi^\pm_{k,\varepsilon,2})| \le C\varepsilon^{\alpha/2}\left(\|\varphi_\varepsilon\|_{L^\infty([0,T];L^\kappa(\Omega))} + \|\mathbf{m}_\varepsilon\|_{L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega))}\right),$$

where

$$\alpha = \min\left\{1-\frac{1}{\kappa}, \frac{1}{2}-\frac{1}{2\gamma}\right\};$$

hence we have to prove only that $b^{\pm}_{k,\varepsilon}(t) = (\phi_\varepsilon(t), \phi^{\pm}_{k,\varepsilon,2})$ converges strongly to 0 in $L^2(0,T)$ when $k \in I$ and study its oscillations when $k \in J$.

Notice that $\phi_\varepsilon(t) = (\varphi_\varepsilon, \mathbf{m}_\varepsilon)$ solves

$$\partial_t \phi_\varepsilon - \frac{L^*_\varepsilon \phi_\varepsilon}{\varepsilon} = \begin{pmatrix} 0 \\ g_\varepsilon \end{pmatrix}, \tag{5.1}$$

where L^*_ε denotes the adjoint of L_ε with respect to $(\cdot,\cdot)$, and

$$g_\varepsilon = -\mathrm{div}(\mathbf{m}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nabla \left[\frac{(\rho_\varepsilon)^\gamma - \gamma\rho_\varepsilon + (\gamma - 1)}{\varepsilon^2} \right] + (\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon.$$

Taking the scalar product of (5.1) with $\phi^{\pm}_{k,\varepsilon,2}$, we obtain

$$\frac{d}{dt} b^{\pm}_{k,\varepsilon}(t) - \frac{\overline{\mathrm{i}\lambda^{\pm}_{k,\varepsilon,2}}}{\varepsilon} b^{\pm}_{k,\varepsilon}(t) = c^{\pm}_{k,\varepsilon}(t), \tag{5.2}$$

where $c^{\pm}_{k,\varepsilon}(t) = (g_\varepsilon, \mathbf{m}^{\pm}_{k,\varepsilon,2}) + \varepsilon^{-1}(\phi_\varepsilon, R^{\pm}_{k,\varepsilon,2})$.

5.2.1. The case *k* ∈ *I*. From (5.2), by Duhamel's principle, we deduce that

$$b^{\pm}_{k,\varepsilon}(t) = b^{\pm}_{k,\varepsilon}(0) \exp^{\overline{\mathrm{i}\lambda^{\pm}_{k,\varepsilon,2}} t/\varepsilon} + \int_0^t c^{\pm}_{k,\varepsilon}(s) \exp^{\overline{\mathrm{i}\lambda^{\pm}_{k,\varepsilon,2}}(t-s)/\varepsilon} ds. \tag{5.3}$$

The first term in (5.3) is estimated as follows:

$$\left\| b^{\pm}_{k,\varepsilon}(0) \exp^{\overline{\mathrm{i}\lambda^{\pm}_{k,\varepsilon,2}} t/\varepsilon} \right\|_{L^2(0,T)} \le C \left\| b^{\pm}_{k,\varepsilon}(0) \exp^{\mathcal{R}e(\overline{\mathrm{i}\lambda^{\pm}_{k,1}}) t/\sqrt{\varepsilon}} \right\|_{L^2(0,T)} \le C\varepsilon^{1/4}.$$

In order to estimate the remaining term in (5.3), we will use the following estimate: for any $1 \le p, q \le \infty$ with $\frac{1}{q} + \frac{1}{p} = 1$, we have

$$\left| \int_0^t \exp^{\overline{\mathrm{i}\lambda^{\pm}_{k,\varepsilon,2}}(t-s)/\varepsilon} a(s) ds \right| \le \int_0^t \exp^{\mathcal{R}e(\overline{\mathrm{i}\lambda^{\pm}_{k,1}})(t-s)/\sqrt{\varepsilon}} |a(s)| ds \le C \|a\|_{L^q(0,T)} \varepsilon^{\frac{1}{2p}}. \tag{5.4}$$

We now write $|c^{\pm}_{k,\varepsilon}| \le c_1 + c_2 + c_3 + c_4$, where

$$\begin{aligned}
c_1(t) &= \left| \int_\Omega (\mathbf{m}_\varepsilon \otimes \mathbf{u}_\varepsilon)(t) \cdot \nabla \mathbf{m}^{\pm}_{k,\varepsilon,2} dx \right|, \\
c_2(t) &= \left| \int_\Omega \left[\frac{(\rho_\varepsilon)^\gamma - \gamma\rho_\varepsilon + (\gamma - 1)}{\varepsilon^2} \right](t) \mathrm{div} \mathbf{m}^{\pm}_{k,\varepsilon,2} dx \right|, \\
c_3(t) &= \varepsilon^{-1} |(\phi_\varepsilon, R^{\pm}_{k,\varepsilon,2})|, \\
c_4(t) &= \left| \int_\Omega (\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon \cdot \nabla \mathbf{m}^{\pm}_{k,\varepsilon,2} dx \right|.
\end{aligned}$$

Observing that $\mathbf{m}_\varepsilon = \varepsilon\varphi_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon$, we have

$$\begin{aligned}
c_1(t) &\le \|\mathbf{m}^{\pm}_{k,\varepsilon,2}\|_{L^\infty(\Omega)} \|\mathbf{u}^1_\varepsilon + \mathbf{u}^2_\varepsilon\|_{L^2(\Omega)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)} \\
&\quad + \varepsilon \|\varphi_\varepsilon\|_{L^\infty([0,T];L^\kappa(\Omega))} \|(u_\varepsilon)^2\|_{L^{\kappa/(\kappa-1)}(\Omega)} \|\nabla \mathbf{m}^{\pm}_{k,\varepsilon,2}\|_{L^\infty(\Omega)} \\
&\le C \|\mathbf{u}^1_\varepsilon\|_{L^\infty([0,T];L^2(\Omega))} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)} + C\varepsilon^{1/2} \|\nabla \mathbf{u}_\varepsilon\|^2_{L^2(\Omega)} + C\varepsilon^{1/2} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)}.
\end{aligned}$$

The second term c_2 is estimated as

$$c_2(t) \leq C\|p_\varepsilon\|_{L^\infty([0,T];L^1(\Omega))}(\|\Psi^\pm_{k,\varepsilon,2}\|_{L^\infty(\Omega)} + \|R^\pm_{k,\varepsilon,2}\|_{L^\infty(\Omega)}) \leq C.$$

Also we estimate c_3 by

$$c_3(t) \leq \frac{1}{\varepsilon}\|R^\pm_{k,\varepsilon,2}\|_{L^{\kappa/(\kappa-1)}(\Omega)}\|\phi_\varepsilon\|_{L^\infty([0,T];L^\kappa(\Omega))} \leq C\varepsilon^{1/2-1/2\kappa}.$$

Finally, we can estimate c_4 by

$$c_4(t) \leq \|\nabla \mathbf{m}^\pm_{k,\varepsilon,2}\|_{L^\infty(\Omega)}\|\mathbf{H}_\varepsilon\|_{L^\infty([0,T];L^2(\Omega))}\|\nabla\mathbf{H}_\varepsilon\|_{L^2(\Omega)} \leq C\|\nabla\mathbf{H}_\varepsilon\|_{L^2(\Omega)}.$$

Therefore, using the estimate (5.4) repeatedly, we can conclude that $b^\pm_{k,\varepsilon}$ converges strongly to 0 in $L^2(0,T)$.

5.2.2. The case *k* ∈ *J*. From (5.3) and the fact that $\lambda^\pm_{k,1} = 0$, we see that $\exp^{\pm i\lambda_{k,0}t/\varepsilon} b^\pm_{k,\varepsilon}$ is bounded in $L^2(0,T)$ and that its time derivative is bounded in $\sqrt{\varepsilon}L^1(0,T) + L^p(0,T)$ for some $p > 1$. It follows that, up to a subsequence, it converges strongly in $L^2(0,T)$ to some element $b^\pm_{k,osc}$.

Next, since $\rho_\varepsilon(x,t)$ converges to 1 in $C([0,T];L^\gamma(\Omega))$ and $b^\pm_{k,\varepsilon}(t)$ are uniformly bounded in $L^2([0,T])$, we deduce that

$$\rho_\varepsilon b^\pm_{k,\varepsilon} b^\pm_{l,\varepsilon}\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}} - b^\pm_{k,\varepsilon} b^\pm_{l,\varepsilon}\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}} \to 0$$

in the sense of distributions. Hence, we need only consider the terms

$$b^\pm_{k,\varepsilon} b^\pm_{l,\varepsilon}\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}} \qquad \text{for all } k, l \in J.$$

On the other hand, due to the strong convergence of $\exp^{\pm i\lambda_{k,0}t/\varepsilon} b^\pm_{k,\varepsilon}$ in $L^2([0,T])$ when $k \in J$, we can deduce that

$$\begin{aligned} &b^\pm_{k,\varepsilon} b^\pm_{l,\varepsilon}\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\Psi_{l,0}}{\lambda_{l,0}} \\ &\quad = \exp^{i(\lambda_{k,0}-\lambda_{l,0})t/\varepsilon}\exp^{-i\lambda_{k,0}t/\varepsilon} b_{k,0}\exp^{i\lambda_{l,0}t/\varepsilon} b_{l,0}\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}} \\ &\quad \to \exp^{i(\lambda_{k,0}-\lambda_{l,0})t/\varepsilon} b^\pm_{k,osc}(t) b^\pm_{l,osc}(t)\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}}, \end{aligned}$$

at least in the sense of distributions. Thus, we are left only to study the interaction of terms

$$\exp^{i(\lambda_{k,0}-\lambda_{l,0})t/\varepsilon} b^\pm_{k,osc}(t) b^\pm_{l,osc}(t)\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}}. \tag{5.5}$$

We will finish the analysis of the interaction by two cases. The first case is $\lambda_{k,0} = \lambda_{l,0}$. In this case, the term (5.5) is reduced to

$$b^\pm_{k,osc}(t) b^\pm_{l,osc}(t)\frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}},$$

whose divergence is clearly a gradient in the sense of distributions, due to the fact that as long as $\lambda_{k,0} = \lambda_{l,0}$, we have

$$\operatorname{div}(\nabla\Psi_{k,0} \otimes \nabla\Psi_{l,0} + \nabla\Psi_{l,0} \otimes \nabla\Psi_{k,0}) = -\lambda_{k,0}^2 \nabla(\Psi_{k,0}\Psi_{l,0}) + \nabla(\nabla\Psi_{k,0} \cdot \nabla\Psi_{l,0}).$$

For the second case, we have $\lambda_{k,0} \neq \lambda_{l,0}$. Under this situation, due to the fact that $b_{k,osc}^{\pm}(t) \in L^2([0,T])$ as $k \in J$, we know that $b_{k,osc}^{\pm}(t) b_{l,osc}^{\pm}(t) \in L^1([0,T])$ for all $k, l \in J$. Then, by the Riemann–Lebesgue lemma, we conclude that

$$\int_0^T \exp^{\mathrm{i}(\lambda_{k,0}-\lambda_{l,0})t/\varepsilon} b_{k,osc}^{\pm}(t) b_{l,osc}^{\pm}(t)dt \to 0 \quad \text{as } \varepsilon \to 0,$$

which implies that (5.5) converges to 0 in the sense of distributions. Hence, the finite sum, as $k, l \leq M$,

$$\operatorname{div}\left(\sum_{k,l\in J} \rho_\varepsilon b_{k,\varepsilon}^{\pm} b_{l,\varepsilon}^{\pm} \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \otimes \frac{\nabla\Psi_{l,0}}{\lambda_{l,0}}\right)$$

converges to a gradient in the sense of distributions. And hence

$$\operatorname{div}(\rho_\varepsilon Q_2\mathbf{u}_\varepsilon \otimes Q_2\mathbf{u}_\varepsilon)$$

converges to a gradient in the sense of distributions.

This completes our proof of Theorem 2.3. ☐

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