MATH 0420 - Review 2

1. Section 5.1: 6, 7, 12.

2. If \( f(x) \) is continuous on an interval \((a, b)\) and \( f(x) > 0 \) for all \( x \in (a, b) \), prove that \( g(x) = \frac{1}{f(x)} \) is continuous on \((a, b)\).

3. If \( f(x) \) and \( g(x) \) are continuous on a set \( A \), show that

\[
H(x) = \max\{f(x), g(x)\}, \quad \text{and} \quad h(x) = \min\{f(x), g(x)\}
\]

are also continuous on \( A \).

4. Section 5.2: 6, 10.

5. A function \( f : \mathbb{R} \to \mathbb{R} \) satisfies the following property: \( f(x + y) = f(x) + f(y) \), \( \forall x, y \in \mathbb{R} \). Suppose that \( f \) is continuous at \( x = 0 \). Prove
   (a) \( f \) is continuous everywhere in \( \mathbb{R} \).
   (b) \( f = f(1)x \), \( \forall x \in \mathbb{R} \).

6. Suppose that \( f : [0, 1] \to \mathbb{R} \) is continuous on the interval \([0, 1]\) and there exists a positive constant \( \alpha \leq 1 \) such that for each \( x \in [0, 1] \) there exists \( y \in [0, 1] \) satisfying \( |f(y)| \leq \alpha |f(x)| \). Prove there exists a point \( c \in [0, 1] \) such that \( f(c) = 0 \).

7. If \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( \lim_{|x| \to \infty} f(x) = 0 \), prove \( f \) is bounded on \( \mathbb{R} \).

8. Show that \( f(x) = \frac{1}{x} \) is not uniformly continuous on \((0, \infty)\) but uniformly continuous on \([1, \infty)\).

9. Suppose \( f : [0, 2] \to \mathbb{R} \) is increasing on \([0, 2]\) and there exist two sequences \( \{a_n\} \) and \( \{b_n\} \) such that \( a_n < 1, b_n > 1, \forall n = 1, 2, 3, \cdots \),

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 1, \quad \text{and} \quad \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} f(b_n) = f(1).
\]

Prove \( f \) is continuous at \( x = 1 \).