Math 0280 *Introduction to Matrices and Linear Algebra*

Exploration Assignment #3
(Matrix Operations)

Acknowledgment *The MATLAB assignments fot Math 0280 were developed by Jonathan Rubin with the help of Matthew Badger*

SCHEDULE: The exploration assignment is due within two weeks. One submission per group (no more than three people) is sufficient. Every member of the group assumes full responsibility for the final product and should be prepared to answer any questions related to the work submitted.

PLAN: In this third assignment, we explore applications of matrices to geometry in the plane. After reviewing how to find the determinant, eigenvalues, and eigenvectors of a matrix in MATLAB, we focus on the applications (1) of determinants to the area of polygons, and (2) of eigenvalues to the dimensions of ellipses.

MATLAB CONCEPTS: Suppose that A is an $n \times n$ square matrix—for this example, say

$$A = \begin{bmatrix} 3 & 3 \\ 5 & 1 \end{bmatrix}.$$

1. To find the determinant of A in MATLAB, type

```
PROMPT>> A=[3,3;5,1]; <enter>
PROMPT>> det(A) <enter>
```

to output the answer:

```
ans =
   -12
```

2. To find the eigenvalues of A, type

```
PROMPT>> eig(A) <enter>
```

to display the eigenvalues of A in a vector:

```
ans =
     6
    -2
```

3. To find the corresponding eigenvectors, type

```
PROMPT>> [V,D]=eig(A) <enter>
```

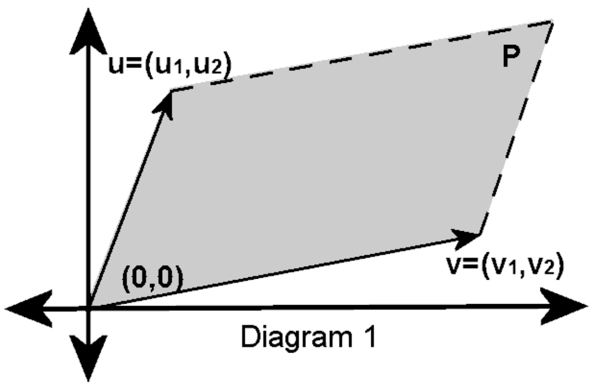
to output two matrices:

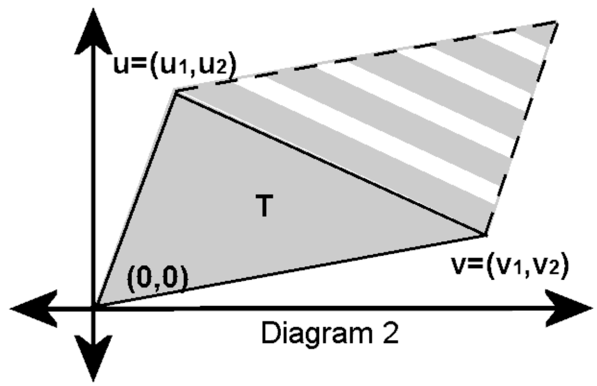
```
V =
    0.7071   -0.5145
    0.7071    0.8575
D =
     6    0
     0   -2
```

The eigenvectors of A appear as the columns of V. And the corresponding eigenvalues appear in the diagonal entries of D. For instance, the eigenvalue $\lambda = 6$ in the (1,1)-entry of D is associated to an eigenvector $\vec{v}$ in the first column of V,

$$\vec{v} \approx \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

<u>GEOMETRY OF DETERMINANTS</u>: (See pp. 283–288 in Poole.) Of the many applications of determinants, one of the most surprising is that determinants give nice formulas for the area of simple geometric shapes. For this assignment, we shall restrict our attention to the two dimensional case.





1. Any two vectors $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$ extending from the origin forms a parallelogram P; this is shown in Diagram 1 above. The area of P is given by

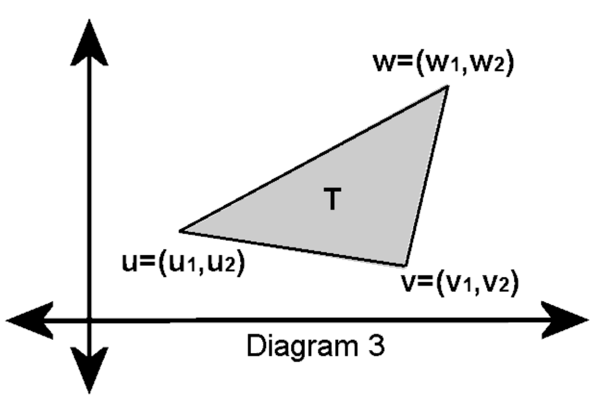
$$\text{Area(P)} = \left| \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|.$$

2. A triangle T is formed in Diagram 2 by drawing a line from the tip of $\vec{u}$ to the tip of $\vec{v}$, dividing P in half. Hence, any triangle T with vertices (u_1, u_2), (v_1, v_2), and $(0, 0)$, satisfies

$$\text{Area}(T) = \frac{1}{2} \left| \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|. \qquad (\heartsuit)$$

In general, a triangle does not need to have a vertex on the origin.

3. Now let T be an arbitrary triangle, say with vertices (u_1, u_2), (v_1, v_2), and (w_1, w_2).



The area of T can be calculated using a 3×3 determinant:

$$\text{Area}(T) = \frac{1}{2}\left|\det \begin{bmatrix} u_1 & u_2 & 1 \\ v_1 & v_2 & 1 \\ w_1 & w_2 & 1 \end{bmatrix}\right|. \qquad (\diamondsuit)$$

Notice that if $(w_1, w_2) = (0, 0)$, then the formula in $(\diamondsuit)$ reduces to the formula in $(\heartsuit)$. Indeed, expanding along the third row,

$$\text{Area}(T) = \frac{1}{2}\left|0 \cdot \det \begin{bmatrix} u_2 & 1 \\ v_2 & 1 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} u_1 & 1 \\ v_1 & 1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right| = \frac{1}{2}\left|\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|.$$

GEOMETRY OF EIGENVALUES:

1. Let A be a symmetric 2×2 matrix. By definition, A is said to be *positive definite* if both of its eigenvalues are positive. Let's look at some examples. Find the eigenvalues of:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Using MATLAB, we find

```
PROMPT>> eig([2,1;1,2]) <enter>
ans =
      1
      3
```

```
PROMPT>> eig([2,1;1,2]) <enter>
ans =
     -1
      3
```

that both of the eigenvalues of A are positive, but one of the eigenvalues of B is negative. Hence, the example A is positive definite, but B is not positive definite.

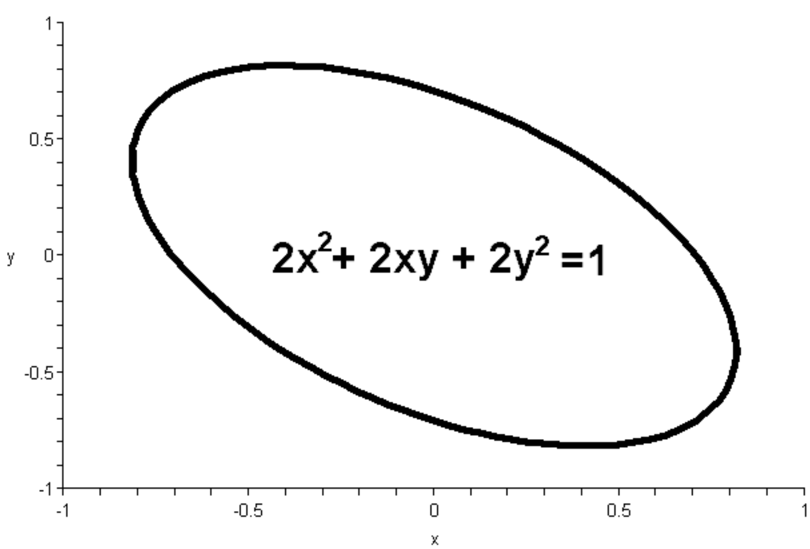
2. Let A be any symmetric, positive definite 2×2 matrix. Then, the equation

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 1 \qquad (\clubsuit)$$

defines an ellipse E_A in the xy-plane. For example, using A from above,

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \qquad \longleftrightarrow \qquad 2x^2 + 2xy + 2y^2 = 1$$

In part III of the assignment, you'll see how the eigenvalues of A govern the geometry of E_A.



<u>ASSIGNMENT</u>: (OUT OF 40 POINTS) Complete all three parts and submit "explore3.m". For each question, please provide any MATLAB commands that you use for computation. The bonus problems are optional and worth 2 points each.

I. Exercise (8 POINTS)

Let A, P, Q be the following 4×4 matrices:

$$A = \begin{bmatrix} 263 & 285 & 310 & 330 \\ -134 & -146 & -158 & -168 \\ 274 & 299 & 321 & 342 \\ -342 & -372 & -402 & -428 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 4 & 5 & 5 & 5 \\ 6 & 6 & 6 & 7 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & 0 & -1 & 0 \\ -2 & -1 & 1 & 0 \\ 4 & 1 & 0 & -1 \\ -6 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the determinant of A? What are the eigenvalues of A?
(b) Fill in the blank: P is the ________ of Q. [*Hint: What is PQ?*]
(c) Let $T = PAQ$. What is the determinant of T? What are the eigenvalues of T?
(d) Fill in the blanks: Let T be an $n \times n$ triangular matrix. The diagonal entries of T are the ________ of T; and, the product of the diagonal entries is the ________ of T.

II. Determinants and Area (16 POINTS)

In (a)–(e), use determinants to find the area of the specified shape.

(a) The parallelogram formed by the vectors $[1, 2]$ and $[-3, 7]$.
(b) The triangle with vertices $(-2, -3)$, $(4, -2)$, $(0, 0)$.
(c) The triangle with vertices $(-2, -3)$, $(4, -2)$, (1,1).
(d) The pentagon with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(1, 3)$, $(0, 2)$.
[*Hint: Divide the pentagon into two pieces, a parallelogram and a triangle.]*
(e) The hexagon with vertices $(1, 0)$, $(2, 1)$, $(2, 2)$, $(1, 3)$, $(0, 2)$, $(0, 1)$.
[*Hint: Sketch the hexagon on top of the pentagon in part (d).*]
(f) Show that the "triangle" with vertices $(2, 3)$, $(-3, 6)$, $(7, 0)$ has an "area" equal to 0. What does this mean geometrically?
($\star$) *Bonus Problem: Using determinants, find the area of the five-pointed star formed by the closed path* $(0, 0)$, $(2, 2)$, $(0, 2)$, $(2, 0)$, $(1, 3)$, $(0, 0)$.

III. Eigenvalues and Ellipses (16 POINTS)

Let X, Y, Z be the following symmetric 2×2 matrices:

$$X = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \qquad Y = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad Z = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

(a) Which of the matrices X, Y and Z are positive definite?
(b) Pick a positive definite matrix from part (a), call it W. Define an ellipse E_W by

$$\begin{bmatrix} x & y \end{bmatrix} W \begin{bmatrix} x \\ y \end{bmatrix} = 1 \qquad (\clubsuit)$$

Write down the equation for E_W in the form $ax^2 + bxy + cy^2 = 1$.

(c) Since $b \neq 0$, the major axis and minor axis of E_W do not lie on the x and y axes. Let's correct this. Make the following substitution in (♣):

$$x = \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}} \qquad \text{and} \qquad y = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}.$$

Write down the equation for E_W in the form $du^2 + ev^2 = 1$.

(d) Geometrically, a substitution of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

is equivalent to rotating the uv-plane counterclockwise by θ. What was θ in part (c)?

(e) Using the equation in part (c), find the lengths of the major and minor axes of E_W.
[*Reference:* "Parametrisation" at `<http://en.wikipedia.org/wiki/Ellipse>`.]

(f) Let $\lambda_1 < \lambda_2$ be the eigenvalues of W. Let ℓ_1 be the length of the major axis of E_W. Write ℓ_1 in terms of λ_1. (In other words, find a function $f(x)$ such that $\ell_1 = f(\lambda_1)$.)
[*Hint: Use the eigenvalues found in part (a) and your answer from part (e).*]

(g) Let ℓ_2 be the length of the minor axis of E_W. Express ℓ_2 in terms of λ_2.

($\star$) *Bonus Problem: Plot E_W in MATLAB.*