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LOCAL STRONG SOLUTION TO THE COMPRESSIBLE MAGNETOHYDRODYNAMIC FLOW WITH LARGE DATA

XIAOLI LI[*,†,‡], NING SU[*,§] and DEHUA WANG[†,¶]

[*]*Department of Mathematical Sciences*
Tsinghua University, Beijing 100084, China
[†]*Department of Mathematics, University of Pittsburgh*
Pittsburgh, PA 15260, USA
[‡]*xllithu@gmail.com*
[§]*nsu@math.tsinghua.edu.cn*
[¶]*dwang@math.pitt.edu*

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Abstract. The three-dimensional compressible magnetohydrodynamic isentropic flow with zero magnetic diffusivity is studied in this paper. The vanishing magnetic diffusivity causes significant difficulties due to the loss of dissipation of the magnetic field. The existence and uniqueness of local-in-time strong solutions with large initial data is established. Strong solutions have weaker regularity than classical solutions. A generalized Lax–Milgram theorem and a Schauder–Tychonoff-type fixed-point argument are applied on conjunction with novel techniques and estimates for strong solutions.

Keywords: Magnetohydrodynamics; zero magnetic diffusivity; strong solutions; existence and uniqueness.

Mathematics Subject Classification 2010: 35Q36, 35D05, 76W05

1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids, such as gases, in an electromagnetic field. If a conducting fluid moves in a magnetic field, electric fields are induced and an electric current flow is developed. The magnetic field exerts forces on these currents which considerably modify the hydrodynamic motion of the fluid. On the other hand, the development of electric currents yields a change in the magnetic field. There is a complex interaction between the magnetic field and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. The equations for compressible magnetohydrodynamics consist of the Euler equations of gas dynamics coupled with the Maxwell's equations of electromagnetic field. The applications of magnetohydrodynamics cover a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics.

The equations of three-dimensional compressible magnetohydrodynamic flow in the isentropic case have the following form (see [2, 22, 23]):

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.1a}$$

$$(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = (\nabla\times\mathbf{H})\times\mathbf{H} + \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}), \tag{1.1b}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = 0, \quad \nabla\cdot\mathbf{H} = 0, \tag{1.1c}$$

where $\rho = \rho(x,t) \in \mathbb{R}^+$ denotes the density, $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^3$ the velocity field, $\mathbf{H} = \mathbf{H}(x,t) \in \mathbb{R}^3$ the magnetic field, and $P(\rho) = A\rho^\gamma$ the pressure with a constant $A > 0$ and the adiabatic exponent $\gamma > 1$. The viscosity coefficients μ and λ of the flow are constants satisfying $\mu > 0$ and $2\mu+3\lambda > 0$, which ensures that the operator $-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})$ is a strictly elliptic operator. The symbol $\otimes$ denotes the usual Kronecker tensor product. Usually, we refer to (1.1a) as the continuity equation (mass conservation equation), and (1.1b) as the momentum conservation equation. It is well known that the electromagnetic fields are governed by the Maxwell's equations. In magnetohydrodynamics, the displacement current can be neglected [22, 23]. As a consequence, Eq. (1.1c) is called the induction equation. As for the constraint $\nabla\cdot\mathbf{H} = 0$, it can be seen just as a restriction on the initial value of $\mathbf{H}$ since $(\nabla\cdot\mathbf{H})_t = 0$. We remark that, the magnetic diffusivity in (1.1) is zero, which arises in the physics regime with negligible electrical resistance, see [5].

We consider the Cauchy problem of (1.1) with the initial condition:

$$(\rho,\mathbf{u},\mathbf{H})(x,0) = (\rho_0,\mathbf{u}_0,\mathbf{H}_0)(x), \quad x\in\mathbb{R}^3, \tag{1.2}$$

and are interested in the existence of solutions to (1.1)–(1.2). When the magnetic diffusivity $\nu \neq 0$, (1.1c) is

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \quad \nabla\cdot\mathbf{H} = 0,$$

and there have been many studies and rich results in the literature: see [3, 4, 6, 8, 9, 14, 18, 19, 23, 33] and the references therein. When the magnetic diffusivity $\nu = 0$ as in (1.1), the mathematical analysis becomes much more difficult due to the loss of dissipation of the magnetic field, and to our best knowledge there have been no results on existence of solutions (even in the incompressible case). The aim of this paper is to establish the local existence and uniqueness of strong solution to system (1.1) with large initial data in the three-dimensional space $\mathbb{R}^3$. By a strong solution, we mean a triplet $(\rho,\mathbf{u},\mathbf{H})$ with $\mathbf{u}(\cdot,t)\in W^{2,q}(\mathbb{R}^3)$ and $(\rho(\cdot,t),\mathbf{H}(\cdot,t))\in W^{1,q}(\mathbb{R}^3)$, $3 < q \le 6$ satisfying (1.1) almost everywhere with the initial condition (1.2). As for the global existence of classical solutions of the small perturbation near an equilibrium for compressible Navier–Stokes equations, we refer the interested reader to [28, 29] and the references cited therein. The global existence of strong solutions with small perturbations near an equilibrium for compressible Navier–Stokes equations was also discussed in [30, 32]. Also see the discussions and references in [30, 32] for other related results on strong solutions.

Throughout this paper, the standard notations for Sobolev spaces $W^{s,p}(\mathbb{R}^3)$ ($H^s(\mathbb{R}^3)$, when $p = 2$) will be used. For $p \in [1, +\infty]$, we denote by $L^p(0, T; X)$ the set of Bochner measurable X-valued time-dependent functions φ such that $t \mapsto \|\varphi\|_X$ belongs to $L^p(0, T)$, and the corresponding Lebesgue norm is defne by $\|\cdot\|_{L^p_T(X)}$. Define the Sobolev space $W^{1,p}(0, T; X) := \{\varphi \mid \varphi \in L^p(0, T; X),\ \varphi_t \in L^p(0, T; X)\}$, and $H^1(0, T; X) := W^{1,2}(0, T; X)$. Precisely, we will establish the following result on existence and uniqueness in this paper.

Theorem 1.1. *Assume that*

$$\rho_0 \in W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \quad \mathbf{u}_0 \in (H^2(\mathbb{R}^3))^3, \quad \mathbf{H}_0 \in (W^{1,q}(\mathbb{R}^3))^3 \cap (H^1(\mathbb{R}^3))^3$$

for some $q \in (3, 6]$, *and*

$$\alpha \leq \rho_0 \leq \beta, \quad |\rho_0\|_{W^{1,q}\cap H^1} + \|\mathbf{u}_0\|_{H^2} + \|\mathbf{H}_0\|_{W^{1,q}\cap H^1} \leq r_0$$

for some positive constants α, β, *and* r_0. *Then there exist positive constants* $\overline{T} = \overline{T}(r_0), \alpha_1(\overline{T}, r_0, \alpha)$, *and* $\beta_1(\overline{T}, r_0, \beta)$, *such that the Cauchy problem* (1.1)–(1.2) *has a unique strong solution* $(\rho, \mathbf{u}, \mathbf{H})$ *on* $\mathbb{R}^3 \times (0, \overline{T})$ *satisfying*

$$\rho \in L^\infty(0, \overline{T}; W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)), \quad \rho_t \in L^\infty(0, \overline{T}; L^q(\mathbb{R}^3)),$$
$$\alpha_1(\overline{T}, r_0, \alpha) \leq \rho \leq \beta_1(\overline{T}, r_0, \beta),$$
$$\mathbf{u} \in (L^2(0, \overline{T}; W^{2,q}(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)))^3, \quad \mathbf{u}_t \in (L^2(0, \overline{T}; H^1(\mathbb{R}^3)))^3,$$
$$\mathbf{H} \in (L^\infty(0, \overline{T};\ W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)))^3, \quad \mathbf{H}_t \in (L^\infty(0, \overline{T}; L^q(\mathbb{R}^3)))^3.$$

In addition to the difficulties due to the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow of large oscillation, another major difficulty in proving the existence is the lack of the dissipative estimates for the magnetic field and the gradient of the density. Since we are concerned with the *strong solutions in* $W^{2,q}$ which have weaker regularity than the *classical solution in* H^3, we need some new techniques and estimates to establish the existence. We first linearize (1.1), use the Lax–Milgram theorem to obtain the solution to the linearized system, then apply the Schauder–Tychonoff fixed-point theorem to obtain the strong solution of (1.1) with large data.

This paper is organized as follows. Section 2, which is the main body of this paper, is devoted to proving the local existence of the system (1.1) by the Lax–Milgram theorem and a fixed-point argument. Section 3 will focus on the uniqueness of the solution obtained in Sec. 2.

2. Local Existence

We use the letter C to denote any constant that can be explicitly computed in terms of known quantities, and the exact value denoted by C may therefore change from line to line in a given computation. Similarly, ε and δ denote arbitrary positive constants, C_ε and C_δ denote correspondingly positive constants depending on $\frac{1}{\varepsilon}$ and $\frac{1}{\delta}$. Set $\partial_i = \partial/\partial x_i,\ i = 1,2,3$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $A = ((a_{ij}))$ and $B = ((b_{ij}))$ are 3×3 matrices, then

$$A : B = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ij} \quad \text{and} \quad |A| = (A : A)^{\frac{1}{2}} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2\right)^{\frac{1}{2}},$$

$$|\mathbf{u}|_p = \text{norm in the space } L^p(\mathbb{R}^3) \text{ with } p \geq 1,$$

$$\|\mathbf{u}\|_s = \text{norm in the Sobolev space } H^s(\mathbb{R}^3) \text{ of order } s \text{ on } L^2(\mathbb{R}^3),$$

$$\|\mathbf{u}\|_{s,p} = \text{norm in the Sobolev space } W^{s,p}(\mathbb{R}^3) \text{ of order } s \text{ on } L^p(\mathbb{R}^3).$$

For a given $T > 0$, define

$$Q_T := \mathbb{R}^3 \times [0, T].$$

Obviously, $L^p(0, T; L^p(\mathbb{R}^3)) = L^p(Q_T)$.

In this section, we will prove the existence result in Theorem 1.1. For simplicity of notations, we set $\mu = \beta = 1$ and $\lambda = 0$ without loss of generality. The proof will proceed through four steps by combining a generalized Lax–Milgram theorem and a Schauder fixed-point argument. To this end, we consider first an auxiliary problem.

Set

$$\Phi = \{\phi(x,t) \in \mathbb{R}^3 \mid \phi \in (L^2(0,T; H^2(\mathbb{R}^3)))^3,\ \phi_t \in (L^2(Q_T))^3\},$$

with the natural norm $\|\phi\|_\Phi$. And, for $q \in (3, 6]$, define

$$\begin{aligned}\Psi = \Phi \cap \{\phi(x,t) \in \mathbb{R}^3 \mid \phi \in (L^2(0,T; W^{2,q}(\mathbb{R}^3)))^3 \cap (L^\infty(0,T; H^2(\mathbb{R}^3)))^3\} \\ \cap \{\phi(x,t) \in \mathbb{R}^3 \mid \phi \in (L^2(0,T; H^2(\mathbb{R}^3)))^3,\ \phi_t \in (L^\infty(0,T; L^2(\mathbb{R}^3)))^3, \\ \nabla\phi_t \in (L^2(Q_T))^9, \phi(0) = \mathbf{u}_0\}.\end{aligned}$$

Using the continuity equation (1.1a), the momentum equation (1.1b) can be reduced to

$$\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla P = (\nabla \times \mathbf{H}) \times \mathbf{H} + \triangle\mathbf{u} + \nabla(\nabla \cdot \mathbf{u}),$$

where the notation $\mathbf{u} \cdot \nabla\mathbf{u}$ is understood to be $(\mathbf{u} \cdot \nabla)\mathbf{u}$. Meanwhile, under the constraint $\nabla \cdot \mathbf{H} = 0$, the induction equation is equivalent to

$$\mathbf{H}_t + \mathbf{u} \cdot \nabla\mathbf{H} = (\nabla\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{I})\ \mathbf{H},$$

where $\mathbf{I}$ stands for the 3×3 identity matrix.

We need to find a triplet $(\rho, \mathbf{u}, \mathbf{H})$ that satisfies the following auxiliary problem:

$$\rho_t + \nabla \cdot (\rho \bar{\mathbf{u}}) = 0, \tag{2.1a}$$

$$\rho \mathbf{u}_t - \triangle \mathbf{u} - \nabla(\nabla \cdot \mathbf{u}) = -\rho \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} - \nabla P + (\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{2.1b}$$

$$\mathbf{H}_t + \bar{\mathbf{u}} \cdot \nabla \mathbf{H} = (\nabla \bar{\mathbf{u}} - (\nabla \cdot \bar{\mathbf{u}})\mathbf{I})\mathbf{H}, \nabla \cdot \mathbf{H} = 0, \tag{2.1c}$$

a.e. in Q_T for any given $T > 0$, with the initial condition (1.2) such that $\mathbf{u} \in \Psi$,

$$\rho \in L^\infty(0,T; W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)) \cap H^1(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)),$$
$$\alpha_1 \le \rho \le \beta_1 \quad (\alpha_1, \beta_1 \text{ are suitable positive constants}),$$

and

$$\mathbf{H} \in (L^\infty(0,T; W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)))^3 \cap (H^1(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)))^3.$$

where $\bar{\mathbf{u}} \in \Psi$, and $\mathbf{u}_0 \in (H^2(\mathbb{R}^3))^3$, $\rho_0 \in W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ with $\alpha \le \rho_0 \le 1$, $\mathbf{H}_0 \in (W^{1,q}(\mathbb{R}^3))^3 \cap (H^1(\mathbb{R}^3))^3$ are given functions and $q \in (3,6]$.

2.1. *Solvability of the density with a fixed velocity*

Obviously, the existence of a unique solution $\rho := \rho(\bar{\mathbf{u}})$ of the continuity equation follows directly from the method of characteristics. Although this method requires that $\bar{\mathbf{u}} \in (C^0(0,T; C^1(\mathbb{R}^3)))^3$ and $\rho_0 \in C^1(\mathbb{R}^3)$, the estimates below in Lemma 2.1 hold under the above assumptions on $\bar{\mathbf{u}}$ and ρ_0, i.e. $\bar{\mathbf{u}} \in \Psi$, $\rho_0 \in W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. So we use, for simplicity, a formal approach. The correct procedure is to consider a regularization of $\bar{\mathbf{u}}$ and ρ_0, and then to pass to limit (similar to the argument for $\mathbf{H}$ below, and can also be found in [30, Theorem 9.3]).

Lemma 2.1. *Under the same conditions as in Theorem 1.1, there is a unique strictly positive function*

$$\rho := \rho(\bar{\mathbf{u}}) \in H^1(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \cap L^\infty(0,T; W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)),$$

which satisfies (2.1a). *Moreover, the density satisfies the following estimate*:

$$\|\nabla \rho\|_{L^\infty(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))} \le (\|\rho_0\|_{W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)} + C\|\bar{\mathbf{u}}\|_\Psi \sqrt{T}) \exp(C\|\bar{\mathbf{u}}\|_\Psi \sqrt{T}). \tag{2.2}$$

Proof. Along characteristics

$$\frac{dX}{dt} = \bar{\mathbf{u}}(t, X),$$
$$X(t) = x,$$

Eq. (2.1a) can be rewritten as

$$\frac{d}{dt}\rho(t,X(t)) = -\rho(t,X(t))\nabla\cdot\bar{\mathbf{u}}(t,X(t)).$$

Then, the explicit formula for ρ is

$$\rho(t,x) = \rho_0(X(0))\,\exp\bigg(-\int_0^t \nabla\cdot\bar{\mathbf{u}}(\tau,X(\tau))d\tau\bigg).$$

It follows that

$$\alpha\exp\bigg(-\int_0^t |\nabla\cdot\bar{\mathbf{u}}(\tau,X(\tau))|_\infty d\tau\bigg) \le \rho(t,x) \le \exp\bigg(\int_0^t |\nabla\cdot\bar{\mathbf{u}}(\tau,X(\tau))|_\infty d\tau\bigg). \tag{2.3}$$

Now applying the gradient operator ∇ to (2.1a), and using

$$\nabla(\nabla\rho\cdot\bar{\mathbf{u}}) = \bar{\mathbf{u}}\cdot\nabla(\nabla\rho) + \nabla\bar{\mathbf{u}}\cdot\nabla\rho,$$

where $\nabla\bar{\mathbf{u}}\cdot\nabla\rho = (\nabla\bar{\mathbf{u}})^\top\nabla\rho$, we have

$$\nabla\rho_t + \bar{\mathbf{u}}\cdot\nabla(\nabla\rho) + \nabla\bar{\mathbf{u}}\cdot\nabla\rho + \rho\nabla(\nabla\cdot\bar{\mathbf{u}}) + \nabla\cdot\bar{\mathbf{u}}\nabla\rho = 0. \tag{2.4}$$

Multiplying (2.4) by $|\nabla\rho|^{q-2}\nabla\rho$ with $3 < q \le 6$ and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{q}\frac{d}{dt}|\nabla\rho|_q^q + \int_{\mathbb{R}^3}\bigg(\frac{1}{q}\bar{\mathbf{u}}\cdot\nabla(|\nabla\rho|^q) + |\nabla\rho|^{q-2}\nabla\rho\cdot(\nabla\bar{\mathbf{u}}\cdot\nabla\rho)$$
$$+\rho|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla(\nabla\cdot\bar{\mathbf{u}}) + |\nabla\rho|^q\nabla\cdot\bar{\mathbf{u}}\bigg)dx = 0.$$

Bearing in mind, by the divergence theorem,

$$\int_{\mathbb{R}^3}(\bar{\mathbf{u}}\cdot\nabla(|\nabla\rho|^q) + |\nabla\rho|^q\nabla\cdot\bar{\mathbf{u}})\,dx = \int_{\mathbb{R}^3}\nabla\cdot(|\nabla\rho|^q\bar{\mathbf{u}})dx = 0,$$

we find, by Hölder's inequality,

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}|\nabla\rho|_q^q &\le |\nabla\rho|_q^{q-1}(|\rho|_\infty|\nabla(\nabla\cdot\bar{\mathbf{u}})|_q + |\nabla\rho|_q|\nabla\bar{\mathbf{u}}|_\infty) + \bigg(\frac{1}{q}-1\bigg)\int_{\mathbb{R}^3}|\nabla\rho|^q\nabla\cdot\bar{\mathbf{u}}\,dx\\ &\le |\nabla\rho|_q^{q-1}(|\rho|_\infty|\nabla(\nabla\cdot\bar{\mathbf{u}})|_q + |\nabla\rho|_q|\nabla\bar{\mathbf{u}}|_\infty) + |\nabla\rho|_q^q|\nabla\cdot\bar{\mathbf{u}}|_\infty\\ &\le C|\nabla\rho|_q^q|\nabla\bar{\mathbf{u}}|_\infty + |\nabla\rho|_q^{q-1}|\rho|_\infty|\nabla(\nabla\cdot\bar{\mathbf{u}})|_q.\end{aligned}$$

Since

$$W^{1,q}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3) \quad \text{as } 3 < q \le 6, \tag{2.5}$$

then we have

$$\begin{aligned}\frac{d}{dt}|\nabla\rho|_q &\le C|\nabla\rho|_q|\nabla\bar{\mathbf{u}}|_\infty + |\rho|_\infty|\nabla(\nabla\cdot\bar{\mathbf{u}})|_q\\ &\le C\|\bar{\mathbf{u}}\|_{2,q}|\nabla\rho|_q + |\rho|_\infty|\nabla(\nabla\cdot\bar{\mathbf{u}})|_q,\end{aligned}$$

and, by Gronwall's inequality,

$$|\nabla\rho|_q \leq \exp\left(C\int_0^t \|\bar{\mathbf{u}}\|_{2,q}d\tau\right)(|\nabla\rho_0|_q + \int_0^t |\rho|_\infty|\nabla(\nabla\cdot\bar{\mathbf{u}})|_q\, d\tau\Big). \tag{2.6}$$

Finally, (2.2) follows from (2.6) and Hölder's inequality. The proof is complete. □

2.2. *Solvability of the magnetic field with a fixed velocity*

Due to the hyperbolic structure of (2.1c) in terms of the magnetic field $\mathbf{H} := \mathbf{H}(\bar{\mathbf{u}})$ with the fixed velocity $\bar{\mathbf{u}}$, we can solve $\mathbf{H}$ through the following Lemma 2.2.

Let $A_j(x,t)$, $j = 1,\ldots,n$, be symmetric $m\times m$ matrices in $\mathbb{R}^n\times(0,T)$, $f(x,t)$ and $V_0(x)$ be two m-dimensional vector functions defined in $\mathbb{R}^n\times(0,T)$ and $\mathbb{R}^n$, respectively. For the Cauchy problem of the linear system in $V\in\mathbb{R}^m\times(0,T)$:

$$\begin{aligned} V_t + \textstyle\sum_{i=1}^n A_j(x,t)\partial_j V + B(x,t)V &= f(x,t),\\ V(x,0) &= V_0(x), \end{aligned} \tag{2.7}$$

we have the following lemma.

Lemma 2.2. *Assume that*

$$A_j \in (C(0,T;H^s(\mathbb{R}^n)) \cap C^1(0,T;H^{s-1}(\mathbb{R}^n)))^{m\times m}, \quad j = 1,\ldots,n,$$
$$B \in (C(0,T;H^{s-1}(\mathbb{R}^n)))^{m\times m}, \quad f \in (C(0,T;H^s(\mathbb{R}^n)))^m, \quad V_0 \in (H^s(\mathbb{R}^n))^m,$$

with $s > \frac{n}{2}+1$ being an integer. Then there exists a unique solution to (2.7), *i.e. a function*

$$V \in (C([0,T),H^s(\mathbb{R}^n)) \cap C^1((0,T),H^{s-1}(\mathbb{R}^n)))^m$$

satisfying (2.7) *pointwise* (*i.e. in the classical sense*).

Proof. This lemma is a direct consequence of [30, Theorem 2.16] with $A_0(x,t) = \mathbf{I}$. □

Now, taking advantage of Lemma 2.2, we have the following lemma.

Lemma 2.3. *Under the same conditions as in Theorem 1.1, there is a unique function*

$$\mathbf{H} := \mathbf{H}(\bar{\mathbf{u}}) \in (H^1(0,T;L^q(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)))^3 \cap (L^\infty(0,T;W^{1,q}(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)))^3,$$

which satisfies Eq. (2.1c). *Moreover, the magnetic field satisfies*

$$\|\mathbf{H}\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3)\cap H^1(\mathbb{R}^3))} \leq (\|\mathbf{H}_0\|_{W^{1,q}(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)} + C\|\bar{\mathbf{u}}\|_\Psi\sqrt{T})\exp(C\|\bar{\mathbf{u}}\|_\Psi\sqrt{T}).$$

Proof. Assume that $\bar{\mathbf{u}} \in (C^1(0,T;C_0^\infty(\mathbb{R}^3)))^3$ and $\mathbf{H}_0 \in (C_0^\infty(\mathbb{R}^3))^3$. Rewriting (2.1c) in the component form ($\bar{\mathbf{u}} = (\bar{\mathbf{u}}^{(1)}, \bar{\mathbf{u}}^{(2)}, \bar{\mathbf{u}}^{(3)})$, $\mathbf{H} = (\mathbf{H}^{(1)}, \mathbf{H}^{(2)}, \mathbf{H}^{(3)})$), employing the summation convention on the repeated indices, we have

$$\mathbf{H}_t^{(i)} + \bar{\mathbf{u}}^{(j)}\partial_j \mathbf{H}^{(i)} = (\partial_j \bar{\mathbf{u}}^{(i)} - (\nabla \cdot \bar{\mathbf{u}})\delta_{ij})\ \mathbf{H}^{(j)}, \quad i = 1,2,3.$$

i.e.

$$\mathbf{H}_t + (\bar{\mathbf{u}}^{(j)}\mathbf{I})\ \partial_j \mathbf{H} + ((\nabla \cdot \bar{\mathbf{u}})\mathbf{I} - \nabla \bar{\mathbf{u}})\ \mathbf{H} = 0.$$

Clearly, $A_j(x,t) = \bar{\mathbf{u}}^{(j)}(x,t)\mathbf{I}$, $j = 1,2,3, B(x,t) = (\nabla \cdot \bar{\mathbf{u}})\mathbf{I} - \nabla \bar{\mathbf{u}}$, and $f(x,t) = 0$ satisfy the assumptions in Lemma 2.2, then we get a unique solution

$$\mathbf{H} \in \bigcap_{s=3}^{\infty} \{(C^1(0,T,H^{s-1}(\mathbb{R}^3)) \cap C(0,T;H^s(\mathbb{R}^3)))^3\},$$

which implies, by the Sobolev imbedding theorems,

$$\mathbf{H} \in \bigcap_{k=1}^{\infty} (C^1(0,T;C^k(\mathbb{R}^3)))^3 = (C^1(0,T;C^\infty(\mathbb{R}^3)))^3.$$

For $\bar{\mathbf{u}} \in \Psi$, $\mathbf{H}_0 \in (W^{1,q}(\mathbb{R}^3))^3 \cap (H^1(\mathbb{R}^3))^3$, by an argument of dense set, there are $\{\bar{\mathbf{u}}_n\}_{n=1}^\infty \subset (C^1(0,T;C_0^\infty(\mathbb{R}^3)))^3$, $\{\mathbf{H}_{0n}\}_{n=1}^\infty \subset (C_0^\infty(\mathbb{R}^3))^3$, respectively, such that

$$\bar{\mathbf{u}}_n \to \bar{\mathbf{u}} \quad \text{in } \Psi,$$
$$\mathbf{H}_{0n} \to \mathbf{H}_0 \quad \text{in } (W^{1,q}(\mathbb{R}^3))^3 \cap (H^1(\mathbb{R}^3))^3.$$

Hence,

$$\bar{\mathbf{u}}_n \to \bar{\mathbf{u}} \quad \text{in } (C([0,T] \times B(0,a)))^3,$$

for any $a > 0$ and $B(0,a)$ denotes the ball with radius a and centered at the origin.

Similarly, there are $\{\mathbf{H}_n\}_{n=1}^\infty \subset (C^1(0,T;C^\infty(\mathbb{R}^3)))^3$ satisfying

$$\mathbf{H}_{nt} + \bar{\mathbf{u}}_n \cdot \nabla \mathbf{H}_n = (\nabla \bar{\mathbf{u}}_n - (\nabla \cdot \bar{\mathbf{u}}_n)\mathbf{I})\mathbf{H}_n \tag{2.8}$$

with $\mathbf{H}_n(0) = \mathbf{H}_{0n}$. Multiplying (2.8) by $|\mathbf{H}_n|^{p-2}\mathbf{H}_n$ ($p \geq 2$), integrating over $\mathbb{R}^3$, by integration by parts, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}|\mathbf{H}_n|_p^p &= -\frac{1}{p}\int_{\mathbb{R}^3} \bar{\mathbf{u}}_n \cdot \nabla(|\mathbf{H}_n|^p)\,dx + \int_{\mathbb{R}^3} |\mathbf{H}_n|^{p-2}\mathbf{H}_n \cdot (\nabla \bar{\mathbf{u}}_n - (\nabla \cdot \bar{\mathbf{u}}_n)\mathbf{I})\ \mathbf{H}_n\,dx \\ &= \frac{1}{p}\int_{\mathbb{R}^3} |\mathbf{H}_n|^p \nabla \cdot \bar{\mathbf{u}}_n\,dx + \int_{\mathbb{R}^3} |\mathbf{H}_n|^{p-2}\mathbf{H}_n \cdot (\nabla \bar{\mathbf{u}}_n - (\nabla \cdot \bar{\mathbf{u}}_n)\mathbf{I})\ \mathbf{H}_n\,dx \\ &\leq C|\mathbf{H}_n|_p^p |\nabla \bar{\mathbf{u}}_n|_\infty,\end{aligned}$$

where $\mathbf{H}_n\cdot(\nabla\bar{\mathbf{u}}_n-(\nabla\cdot\bar{\mathbf{u}}_n)\mathbf{I})=\mathbf{H}_n^\top(\nabla\bar{\mathbf{u}}_n-(\nabla\cdot\bar{\mathbf{u}}_n)\mathbf{I})$. Then, by Gronwall's inequality and the imbedding (2.5), we get

$$|\mathbf{H}_n|_p^p \le \exp\left(C\int_0^t |\nabla\bar{\mathbf{u}}_n|_\infty d\tau\right)|\mathbf{H}_n(0)|_p^p \le \exp\left(C\int_0^t \|\bar{\mathbf{u}}_n\|_{2,q}\,d\tau\right)|\mathbf{H}_{0n}|_p^p.$$

i.e.

$$|\mathbf{H}_n|_p \le \exp\left(C\int_0^t \|\bar{\mathbf{u}}_n\|_{2,q}d\tau\right)|\mathbf{H}_{0n}|_p.$$

Thus, by Hölder's inequality, we have

$$\|\mathbf{H}_n\|_{L_T^\infty(L^p(\mathbb{R}^3))} \le \exp(C\ \|\bar{\mathbf{u}}_n\|_\Psi\sqrt{T})|\mathbf{H}_{0n}|_p. \tag{2.9}$$

In particular, if we choose $p=q$ in (2.9), then

$$\|\mathbf{H}_n\|_{L_T^\infty(L^q(\mathbb{R}^3))} \le \exp(C\ \|\bar{\mathbf{u}}_n\|_\Psi\sqrt{T})|\mathbf{H}_{0n}|_q < \infty,$$

and, up to a subsequence, assuming that $\{\mathbf{H}_n\}_{n=1}^\infty$ were chosen so that

$$\mathbf{H}_n \to \mathbf{H} \quad \text{weak-* in } (L^\infty(0,T;L^q(\mathbb{R}^3)))^3.$$

Moreover, letting $p\to\infty$ in (2.9), using the imbedding (2.5) again, we obtain

$$\begin{aligned}\|\mathbf{H}_n\|_{L^\infty(Q_T)} &\le \exp(C\|\bar{\mathbf{u}}_n\|_\Psi\sqrt{T})|\mathbf{H}_{0n}|_\infty\\ &\le C\exp(C\|\bar{\mathbf{u}}_n\|_\Psi\sqrt{T})\|\mathbf{H}_{0n}\|_{1,q} < \infty.\end{aligned}$$

Taking the gradient in both sides of (2.8), multiplying by $|\nabla\mathbf{H}_n|^{q-2}\nabla\mathbf{H}_n$ and integrating over $\mathbb{R}^3$, we get, with the help of Hölder's inequality and the imbedding (2.5),

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}|\nabla\mathbf{H}_n|_q^q &= -\int_{\mathbb{R}^3}|\nabla\mathbf{H}_n|^{q-2}\nabla\mathbf{H}_n : (\nabla(\bar{\mathbf{u}}_n\cdot\nabla\mathbf{H}_n))\,dx\\ &\quad+\int_{\mathbb{R}^3}|\nabla\mathbf{H}_n|^{q-2}\nabla\mathbf{H}_n : (\nabla((\nabla\bar{\mathbf{u}}_n-(\nabla\cdot\bar{\mathbf{u}}_n)\mathbf{I})\mathbf{H}_n))\,dx\\ &= -\int_{\mathbb{R}^3}|\nabla\mathbf{H}_n|^{q-2}\sum_{i=1}^3(\partial_i\mathbf{H}_n\cdot\partial_i(\bar{\mathbf{u}}_n\cdot\nabla\mathbf{H}_n)\\ &\quad-\partial_i\mathbf{H}_n\cdot\partial_i((\nabla\bar{\mathbf{u}}_n-(\nabla\cdot\bar{\mathbf{u}}_n)\mathbf{I})\mathbf{H}_n))\,dx\\ &= -\int_{\mathbb{R}^3}|\nabla\mathbf{H}_n|^{q-2}\sum_{i,j=1}^3(\partial_i\mathbf{H}_n^{(j)}(\partial_i\bar{\mathbf{u}}_n\cdot\nabla\mathbf{H}_n^{(j)}+\bar{\mathbf{u}}_n\cdot\partial_i(\nabla\mathbf{H}_n^{(j)})))\,dx\\ &\quad+\int_{\mathbb{R}^3}|\nabla\mathbf{H}_n|^{q-2}\sum_{i,j,k=1}^3(\partial_i\mathbf{H}_n^{(j)}\partial_i\mathbf{H}_n^{(k)}(\partial_k\bar{\mathbf{u}}_n^{(j)}-(\nabla\cdot\bar{\mathbf{u}}_n)\delta_{jk})\\ &\quad+\partial_i\mathbf{H}_n^{(j)}\mathbf{H}_n^{(k)}\partial_i(\partial_k\bar{\mathbf{u}}_n^{(j)}-(\nabla\cdot\bar{\mathbf{u}}_n)\delta_{jk}))\,dx\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{\mathbb{R}^3}|\nabla \mathbf{H}_n|^q|\nabla \bar{\mathbf{u}}_n|\,dx-\frac{1}{q}\int_{\mathbb{R}^3}\bar{\mathbf{u}}_n\cdot\nabla(|\nabla \mathbf{H}_n|^q)\,dx\\
&\quad +C\int_{\mathbb{R}^3}|\nabla \mathbf{H}_n|^q|\nabla \bar{\mathbf{u}}_n|\,dx+C\int_{\mathbb{R}^3}|\mathbf{H}_n||\nabla \mathbf{H}_n|^{q-1}|\nabla\nabla \bar{\mathbf{u}}_n|\,dx\\
&\le C\int_{\mathbb{R}^3}|\nabla \mathbf{H}_n|^q|\nabla \bar{\mathbf{u}}_n|\,dx+C\int_{\mathbb{R}^3}|\mathbf{H}_n||\nabla \mathbf{H}_n|^{q-1}|\nabla\nabla \bar{\mathbf{u}}_n|\,dx\\
&\le C|\nabla \bar{\mathbf{u}}_n|_\infty|\nabla \mathbf{H}_n|_q^q+C|\mathbf{H}_n|_\infty\|\bar{\mathbf{u}}_n\|_{2,q}|\nabla \mathbf{H}_n|_q^{q-1}\\
&\le C\|\bar{\mathbf{u}}_n\|_{2,q}|\nabla \mathbf{H}_n|_q^q+C\|\bar{\mathbf{u}}_n\|_{2,q}|\nabla \mathbf{H}_n|_q^{q-1}.
\end{aligned}
\tag{2.10}
$$

Using Gronwall's inequality, we conclude that

$$
|\nabla \mathbf{H}_n|_q\le \exp\biggl(C\int_0^t\|\bar{\mathbf{u}}_n\|_{2,q}d\tau\biggr)\biggl(|\nabla \mathbf{H}_n(0)|_q+C\int_0^t\|\bar{\mathbf{u}}_n\|_{2,q}\,d\tau\biggr),
$$

and hence,

$$
|\nabla \mathbf{H}|_q\le \liminf_{n\to\infty}|\nabla \mathbf{H}_n|_q\le \exp\biggl(C\int_0^t\|\bar{\mathbf{u}}\|_{2,q}d\tau\biggr)\biggl(|\nabla \mathbf{H}_0|_q+C\int_0^t\|\bar{\mathbf{u}}\|_{2,q}\,d\tau\biggr).
\tag{2.11}
$$

Furthermore,

$$
\|\mathbf{H}\|_{L_T^\infty(W^{1,q}(\mathbb{R}^3))}\le \exp(C\|\bar{\mathbf{u}}\|_\Psi\sqrt{T})(\|\mathbf{H}_0\|_{1,q}+C\|\bar{\mathbf{u}}\|_\Psi\sqrt{T}).
$$

Passing to the limit as $n\to\infty$ in (2.8), we show that (2.1c) holds at least in the sense of distributions. Therefore, $\mathbf{H}_t\in(L^2(0,T;L^2(\mathbb{R}^3)))^3$, then $\mathbf{H}\in(H^1(0,T;L^q(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)))^3$. The proof is complete. □

2.3. *Local solvability of* (2.1b)

Now we prove the existence of a solution of (2.1b). First we consider the bilinear form $E(\mathbf{u},\phi)$ and the functional $L(\phi)$ defined by

$$
\begin{aligned}
E(\mathbf{u},\phi)&=\int_0^T(\rho\mathbf{u}_t-\triangle\mathbf{u}-\nabla(\nabla\cdot\mathbf{u}),\ \phi_t-k(\triangle\phi+\nabla(\nabla\cdot\phi)))\,dt\\
&\quad-(\mathbf{u}(0),\triangle\phi(0)+\nabla(\nabla\cdot\phi(0))),\\
L(\phi)&=-\int_0^T(\rho\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}+\nabla P-(\nabla\times\mathbf{H})\times\mathbf{H},\ \phi_t-k(\triangle\phi+\nabla(\nabla\cdot\phi)))\,dt\\
&\quad-(\mathbf{u}_0,\triangle\phi(0)+\nabla(\nabla\cdot\phi(0)))
\end{aligned}
$$

with $k=(\|\rho\|_{L^\infty(Q_T)})^{-1}$ for $\phi\in\Phi$. Here, and in what follows, $(\cdot,\cdot)$ denotes the inner product in $(L^2(\mathbb{R}^3))^3$. Obviously, $L(\phi)$ is linear continuous on Φ with respect to the norm $\|\phi\|_\Phi$. Moreover, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}E(\phi,\phi) &= \int_0^T (|\sqrt{\rho}\phi_t|_2^2 + k|\triangle\phi + \nabla(\nabla\cdot\phi)|_2^2 - k(\rho\phi_t, \triangle\phi + \nabla(\nabla\cdot\phi))\,dt \\ &\quad + \frac{1}{2}(|\nabla\phi(T)|_2^2 + |\nabla\phi(0)|_2^2 + |\nabla\cdot\phi(T)|_2^2 + |\nabla\cdot\phi(0)|_2^2) \\ &\geq \int_0^T \left(|\sqrt{\rho}\phi_t|_2^2 + k|\triangle\phi + \nabla(\nabla\cdot\phi)|_2^2 - \frac{3}{4}|\sqrt{\rho}\phi_t|_2^2 - \frac{k}{3}|\triangle\phi + \nabla(\nabla\cdot\phi)|_2^2\right)dt \\ &\quad + \frac{1}{2}(|\nabla\phi(T)|_2^2 + |\nabla\phi(0)|_2^2 + |\nabla\cdot\phi(T)|_2^2 + |\nabla\cdot\phi(0)|_2^2) \\ &\geq C\|\phi\|_\Phi^2.\end{aligned}$$

Hence, by the Lax–Milgram theorem (see [15]), there exists a $\mathbf{u} \in \Phi$ such that

$$E(\mathbf{u},\phi) = L(\phi) \tag{2.12}$$

for every $\phi \in \Phi$.

If we assume that $\overline{\phi}$ is a solution of the problem

$$\begin{aligned}\overline{\phi}_t - k(\triangle\overline{\phi} + \nabla(\nabla\cdot\overline{\phi})) &= 0, \\ \overline{\phi}(0) &= h(x)\end{aligned}$$

with $h(x)$ smooth enough, and replace in (2.12) ϕ by $\overline{\phi}$, then we have

$$(\mathbf{u}(0) - \mathbf{u}_0, \triangle h + \nabla(\nabla\cdot h)) = 0,$$

which implies $\mathbf{u}(0) = \mathbf{u}_0$. Similarly, if $\widetilde{\phi}$ is a solution of the problem

$$\begin{aligned}\widetilde{\phi}_t - k(\triangle\widetilde{\phi} + \nabla(\nabla\cdot\widetilde{\phi})) &= g(x,t), \\ \widetilde{\phi}(0) &= 0\end{aligned}$$

with g smooth enough, replacing ϕ by $\widetilde{\phi}$ in (2.12), then we get

$$\int_0^T (\rho\mathbf{u}_t - \triangle\mathbf{u} - \nabla(\nabla\cdot\mathbf{u}) + \rho\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}} + \nabla P - (\nabla\times\mathbf{H})\times\mathbf{H}, g)\,dt = 0,$$

which implies that $(\mathbf{u},\rho,\mathbf{H})$ satisfies (2.1) a.e. in Q_T.

Next, we prove the higher regularity for $\mathbf{u}$. To avoid tedious calculations and notation, we work directly with the derivatives with respect to t of $\mathbf{u}$ instead of its differential quotients. Multiplying (2.1b) by $\mathbf{u}_t$, integrating over Q_t and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\int_0^t |\sqrt{\rho}\mathbf{u}_t|_2^2\,d\tau + \frac{1}{2}|\nabla\mathbf{u}|_2^2 + \frac{1}{2}|\nabla\cdot\mathbf{u}|_2^2 \\ &\quad = \frac{1}{2}|\nabla\mathbf{u}(0)|_2^2 + \frac{1}{2}|\nabla\cdot\mathbf{u}(0)|_2^2 \\ &\qquad + \int_0^t (-\rho\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}} - \nabla P + (\nabla\times\mathbf{H})\times\mathbf{H},\ \mathbf{u}_t)d\tau\end{aligned}$$

$$\le \frac{1}{2}|\nabla \mathbf{u}(0)|_2^2 + \frac{1}{2}|\nabla\cdot \mathbf{u}(0)|_2^2 + \int_0^t \int_{\mathbb{R}^3} \left(\frac{1}{3}|\sqrt{\rho}\mathbf{u}_t|^2 + \frac{3}{4}|\sqrt{\rho}\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}|^2\right) dxd\tau$$
$$+ \int_0^t \int_{\mathbb{R}^3} P\nabla\cdot\mathbf{u}_t\, dxd\tau + \int_0^t (\mathbf{H}\cdot\nabla\mathbf{H},\ \mathbf{u}_t)\, d\tau + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} |\mathbf{H}|^2\nabla\cdot\mathbf{u}_t\, dxd\tau. \tag{2.13}$$

By Hölder's inequality and the Gagliardo–Nirenberg–Sobolev inequalities

$$|\nabla\bar{\mathbf{u}}|_3 \le C|\nabla\bar{\mathbf{u}}|_2^{\frac{1}{2}}|\triangle\bar{\mathbf{u}}|_2^{\frac{1}{2}}, \quad |\bar{\mathbf{u}}|_6 \le C|\nabla\bar{\mathbf{u}}|_2, \tag{2.14}$$

it is easy to deduce from (2.13) that

$$\frac{2}{3}\int_0^t |\sqrt{\rho}\mathbf{u}_t|_2^2\, d\tau + \frac{1}{2}|\nabla\mathbf{u}|_2^2 + \frac{1}{2}|\nabla\cdot\mathbf{u}|_2^2$$
$$\le \frac{1}{2}|\nabla \mathbf{u}(0)|_2^2 + \frac{1}{2}|\nabla\cdot \mathbf{u}(0)|_2^2 + \frac{3}{4}\sup_{0\le\tau\le t}|\rho|_\infty \int_0^t |\bar{\mathbf{u}}|_6^2|\nabla\bar{\mathbf{u}}|_3^2\, d\tau$$
$$+ \int_0^t \int_{\mathbb{R}^3} P\nabla\cdot\mathbf{u}_t\, dxd\tau + \int_0^t (\mathbf{H}\cdot\nabla\mathbf{H},\ \mathbf{u}_t)\, d\tau + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} |\mathbf{H}|^2\nabla\cdot\mathbf{u}_t\, dxd\tau$$
$$\le \frac{1}{2}|\nabla \mathbf{u}(0)|_2^2 + \frac{1}{2}|\nabla\cdot \mathbf{u}(0)|_2^2 + C\sup_{0\le\tau\le t}|\rho|_\infty \int_0^t |\nabla\bar{\mathbf{u}}|_2^3|\triangle\bar{\mathbf{u}}|_2\, d\tau$$
$$+ C_\delta\left(\int_0^t |P|_2^2\, d\tau + \int_0^t \left|\frac{1}{\sqrt{\rho}}\mathbf{H}\right|_\infty^2 |\nabla\mathbf{H}|_2^2\, d\tau + \int_0^t |\mathbf{H}|_\infty^2|\mathbf{H}|_2^2\, d\tau\right)$$
$$+ \delta\int_0^t |\sqrt{\rho}\mathbf{u}_t|_2^2\, d\tau + 2\delta\int_0^t |\nabla\cdot\mathbf{u}_t|_2^2\, d\tau. \tag{2.15}$$

Now we differentiate (2.1b) with respect to t and get

$$\rho_t\mathbf{u}_t + \rho\mathbf{u}_{tt} - \triangle\mathbf{u}_t - \nabla(\nabla\cdot\mathbf{u}_t) = -\rho_t\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}$$
$$- \rho\bar{\mathbf{u}}_t\cdot\nabla\bar{\mathbf{u}} - \rho\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}_t - \nabla P_t + \mathbf{H}_t\cdot\nabla\mathbf{H} + \mathbf{H}\cdot\nabla\mathbf{H}_t - \nabla(\mathbf{H}\cdot\mathbf{H}_t). \tag{2.16}$$

Multiplying (2.16) by $\mathbf{u}_t$, integrating over $\mathbb{R}^3$, bearing in mind the continuity equation (2.1a) and the induction equation (2.1c), we obtain

$$\frac{1}{2}\frac{d}{dt}|\sqrt{\rho}\mathbf{u}_t|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} \rho_t|\mathbf{u}_t|^2\, dx + |\nabla\mathbf{u}_t|_2^2 + |\nabla\cdot\mathbf{u}_t|_2^2$$
$$\le |\rho|_\infty|\bar{\mathbf{u}}|_\infty|\nabla\bar{\mathbf{u}}|_2|\nabla\bar{\mathbf{u}}|_3|\mathbf{u}_t|_6 + |\rho|_\infty|\bar{\mathbf{u}}|_6^2|\nabla(\nabla\bar{\mathbf{u}})|_2|\mathbf{u}_t|_6$$
$$+ |\rho|_\infty|\bar{\mathbf{u}}|_\infty|\bar{\mathbf{u}}|_6|\nabla\bar{\mathbf{u}}|_3|\nabla\mathbf{u}_t|_2$$
$$+ |\sqrt{\rho}|_\infty|\sqrt{\rho}\bar{\mathbf{u}}_t|_2|\nabla\bar{\mathbf{u}}|_3|\mathbf{u}_t|_6 + |\sqrt{\rho}|_\infty|\nabla\bar{\mathbf{u}}_t|_2|\bar{\mathbf{u}}|_\infty|\sqrt{\rho}\mathbf{u}_t|_2 + \|P'\|_{C^0_{\mathrm{loc}}}|\rho_t|_2|\nabla\mathbf{u}_t|_2$$
$$+ (\nabla\times(\bar{\mathbf{u}}\times\mathbf{H})\cdot\nabla\mathbf{H}, \mathbf{u}_t) + (\mathbf{H}\cdot\nabla\mathbf{H}_t,\ \mathbf{u}_t) + |\mathbf{H}|_\infty|\mathbf{H}_t|_2|\nabla\cdot\mathbf{u}_t|_2$$
$$= |\rho|_\infty|\bar{\mathbf{u}}|_\infty|\nabla\bar{\mathbf{u}}|_2|\nabla\bar{\mathbf{u}}|_3|\mathbf{u}_t|_6 + |\rho|_\infty|\bar{\mathbf{u}}|_6^2|\nabla(\nabla\bar{\mathbf{u}})|_2|\mathbf{u}_t|_6$$

$$
\begin{aligned}
&+|\rho|_\infty|\bar{\mathbf{u}}|_\infty|\bar{\mathbf{u}}|_6|\nabla\bar{\mathbf{u}}|_3|\nabla\mathbf{u}_t|_2\\
&+|\sqrt{\rho}|_\infty|\sqrt{\rho}\bar{\mathbf{u}}_t|_2|\nabla\bar{\mathbf{u}}|_3|\mathbf{u}_t|_6+|\sqrt{\rho}|_\infty|\nabla\bar{\mathbf{u}}_t|_2|\bar{\mathbf{u}}|_\infty|\sqrt{\rho}\mathbf{u}_t|_2+\|P'\|_{C^0_{\rm loc}}|\rho_t|_2|\nabla\mathbf{u}_t|_2\\
&-\int_{\mathbb{R}^3}\mathbf{H}\cdot(\nabla\mathbf{u}_t)\nabla\times(\bar{\mathbf{u}}\times\mathbf{H})\,dx-\int_{\mathbb{R}^3}\mathbf{H}_t\cdot(\nabla\mathbf{u}_t)\ \mathbf{H}\,dx+|\mathbf{H}|_\infty|\mathbf{H}_t|_2|\nabla\cdot\mathbf{u}_t|_2\\
\le\ &|\rho|_\infty|\bar{\mathbf{u}}|_\infty|\nabla\bar{\mathbf{u}}|_2|\nabla\bar{\mathbf{u}}|_3|\mathbf{u}_t|_6+|\rho|_\infty|\bar{\mathbf{u}}|_6^2|\nabla(\nabla\bar{\mathbf{u}})|_2|\mathbf{u}_t|_6\\
&+|\rho|_\infty|\bar{\mathbf{u}}|_\infty|\bar{\mathbf{u}}|_6|\nabla\bar{\mathbf{u}}|_3|\nabla\mathbf{u}_t|_2\\
&+|\sqrt{\rho}|_\infty|\sqrt{\rho}\bar{\mathbf{u}}_t|_2|\nabla\bar{\mathbf{u}}|_3|\mathbf{u}_t|_6+|\sqrt{\rho}|_\infty|\nabla\bar{\mathbf{u}}_t|_2|\bar{\mathbf{u}}|_\infty|\sqrt{\rho}\mathbf{u}_t|_2+\|P'\|_{C^0_{\rm loc}}|\rho_t|_2|\nabla\mathbf{u}_t|_2\\
&+|\mathbf{H}|_\infty|\bar{\mathbf{u}}|_\infty|\nabla\mathbf{H}|_2|\nabla\mathbf{u}_t|_2+|\mathbf{H}|_\infty^2|\nabla\bar{\mathbf{u}}|_2|\nabla\mathbf{u}_t|_2+|\mathbf{H}|_\infty^2|\nabla\cdot\bar{\mathbf{u}}|_2|\nabla\mathbf{u}_t|_2\\
&+|\mathbf{H}|_\infty|\mathbf{H}_t|_2|\nabla\mathbf{u}_t|_2+|\mathbf{H}|_\infty|\mathbf{H}_t|_2|\nabla\cdot\mathbf{u}_t|_2. \qquad (2.17)
\end{aligned}
$$

Integrating (2.17) with respect to t, taking advantage of the continuity equation and the Gagliardo–Nirenberg–Sobolev inequalities as (2.14), we find, since $\rho\in L^\infty(Q_T)$,

$$
\begin{aligned}
&\frac{1}{2}|\sqrt{\rho}\mathbf{u}_t|_2^2+\int_0^t(|\nabla\mathbf{u}_t|_2^2+|\nabla\cdot\mathbf{u}_t|_2^2)d\tau\\
&\le\ \frac{1}{2}|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2^2+C\int_0^t|\rho|_\infty(|\bar{\mathbf{u}}|_\infty|\nabla\bar{\mathbf{u}}|_2^{\frac{3}{2}}|\triangle\bar{\mathbf{u}}|_2^{\frac{1}{2}}|\nabla\mathbf{u}_t|_2\\
&\quad+|\nabla\bar{\mathbf{u}}|_2^2|\triangle\bar{\mathbf{u}}|_2|\nabla\mathbf{u}_t|_2)\,d\tau\\
&\quad+C\int_0^t|\sqrt{\rho}|_\infty(|\sqrt{\rho}\bar{\mathbf{u}}_t|_2|\nabla\bar{\mathbf{u}}|_2^{\frac{1}{2}}|\triangle\bar{\mathbf{u}}|_2^{\frac{1}{2}}|\nabla\mathbf{u}_t|_2+|\bar{\mathbf{u}}|_\infty|\nabla\bar{\mathbf{u}}_t|_2|\sqrt{\rho}\mathbf{u}_t|_2)\,d\tau\\
&\quad+\int_0^t\|P'\|_{C^0_{\rm loc}}|\rho_t|_2|\nabla\mathbf{u}_t|_2\,d\tau+\int_0^t(|\mathbf{H}|_\infty|\bar{\mathbf{u}}|_\infty|\nabla\mathbf{H}|_2|\nabla\mathbf{u}_t|_2\\
&\quad+|\mathbf{H}|_\infty^2|\nabla\bar{\mathbf{u}}|_2|\nabla\mathbf{u}_t|_2+|\mathbf{H}|_\infty^2|\nabla\cdot\bar{\mathbf{u}}|_2|\nabla\mathbf{u}_t|_2+|\mathbf{H}|_\infty|\mathbf{H}_t|_2|\nabla\mathbf{u}_t|_2\\
&\quad+|\mathbf{H}|_\infty|\mathbf{H}_t|_2|\nabla\cdot\mathbf{u}_t|_2)\,d\tau\\
&\le\ \frac{1}{2}|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2^2+C_\delta\bigg\{\sup_{0\le\tau\le t}|\rho|_\infty^2\int_0^t(|\bar{\mathbf{u}}|_\infty^2|\nabla\bar{\mathbf{u}}|_2^3|\triangle\bar{\mathbf{u}}|_2+|\nabla\bar{\mathbf{u}}|_2^4|\triangle\bar{\mathbf{u}}|_2^2)\,d\tau\\
&\quad+\sup_{0\le\tau\le t}|\sqrt{\rho}|_\infty^2\int_0^t|\sqrt{\rho}\bar{\mathbf{u}}_t|_2^2|\nabla\bar{\mathbf{u}}|_2|\triangle\bar{\mathbf{u}}|_2\,d\tau+\sup_{0\le\tau\le t}|\sqrt{\rho}|_\infty^2\int_0^t|\bar{\mathbf{u}}|_\infty^2|\sqrt{\rho}\mathbf{u}_t|_2^2\,d\tau\\
&\quad+\int_0^t\|P'\|_{C^0_{\rm loc}}^2|\rho_t|_2^2\,d\tau+\int_0^t(|\mathbf{H}|_\infty^2|\bar{\mathbf{u}}|_\infty^2|\nabla\mathbf{H}|_2^2+|\mathbf{H}|_\infty^4|\nabla\bar{\mathbf{u}}|_2^2+|\mathbf{H}|_\infty^4|\nabla\cdot\bar{\mathbf{u}}|_2^2\\
&\quad+2|\mathbf{H}|_\infty^2|\mathbf{H}_t|_2^2)\,d\tau\bigg\}+8\delta\int_0^t|\nabla\mathbf{u}_t|_2^2\,d\tau+\delta\int_0^t|\nabla\cdot\mathbf{u}_t|_2^2\,d\tau+\delta\int_0^t|\nabla\bar{\mathbf{u}}_t|_2^2\,d\tau, \qquad (2.18)
\end{aligned}
$$

where $\delta>0$ is small enough.

Summing (2.15) and (2.18), and for suitable δ ($\delta \le \frac{1}{16}$), we can first obtain, by Gronwall's inequality,

$$\|\sqrt{\rho}\mathbf{u}_t\|_{L^\infty_T(L^2(\mathbb{R}^3))} \le C,$$

and secondly,

$$\begin{aligned}
&\frac{1}{2}|\sqrt{\rho}\mathbf{u}_t|_2^2 + \frac{1}{2}|\nabla\mathbf{u}|_2^2 + \frac{1}{2}|\nabla\cdot\mathbf{u}|_2^2 + \frac{1}{3}\int_0^t |\sqrt{\rho}\mathbf{u}_t|_2^2\,d\tau + \frac{1}{4}\int_0^t (|\nabla\mathbf{u}_t|_2^2 + |\nabla\cdot\mathbf{u}_t|_2^2)\,d\tau\\
&\quad\le \frac{1}{2}|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2^2 + \frac{1}{2}|\nabla\mathbf{u}(0)|_2^2 + \frac{1}{2}|\nabla\cdot\mathbf{u}(0)|_2^2 + C_\delta\int_0^t |P|_2^2\,d\tau\\
&\qquad + C_\delta\int_0^t \left|\frac{1}{\sqrt{\rho}}\mathbf{H}\right|_\infty^2 |\nabla\mathbf{H}|_2^2\,d\tau + C\sup_{0\le\tau\le t}|\rho|_\infty\int_0^t |\nabla\bar{\mathbf{u}}|_2^3|\triangle\bar{\mathbf{u}}|_2\,d\tau\\
&\qquad + C_\delta\Bigg\{\sup_{0\le\tau\le t}|\rho|_\infty^2\int_0^t (|\bar{\mathbf{u}}|_\infty^2|\nabla\bar{\mathbf{u}}|_2^3|\triangle\bar{\mathbf{u}}|_2 + |\nabla\bar{\mathbf{u}}|_2^4|\triangle\bar{\mathbf{u}}|_2^2)\,d\tau\\
&\qquad + \sup_{0\le\tau\le t}|\sqrt{\rho}|_\infty^2\left(\int_0^t |\sqrt{\rho}\bar{\mathbf{u}}_t|_2^2|\nabla\bar{\mathbf{u}}|_2|\triangle\bar{\mathbf{u}}|_2\,d\tau + \int_0^t |\bar{\mathbf{u}}|_\infty^2|\sqrt{\rho}\mathbf{u}_t|_2^2\,d\tau\right)\\
&\qquad + \int_0^t \|P'\|_{C^0_{\mathrm{loc}}}^2|\rho_t|_2^2\,d\tau + \int_0^t (|\mathbf{H}|_\infty^2|\bar{\mathbf{u}}|_\infty^2|\nabla\mathbf{H}|_2^2 + |\mathbf{H}|_\infty^4|\nabla\bar{\mathbf{u}}|_2^2\\
&\qquad + |\mathbf{H}|_\infty^4|\nabla\cdot\bar{\mathbf{u}}|_2^2 + 2|\mathbf{H}|_\infty^2|\mathbf{H}_t|_2^2 + |\mathbf{H}|_\infty^2|\mathbf{H}|_2^2)\,d\tau\Bigg\} + \delta\int_0^t |\nabla\bar{\mathbf{u}}_t|_2^2\,d\tau,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sqrt{\rho}\mathbf{u}_t \in (L^\infty(0,T;L^2(\mathbb{R}^3)))^3, \quad \mathbf{u}_t \in (L^2(0,T;H^1(\mathbb{R}^3)))^3,\\
&\nabla\mathbf{u} \in (L^\infty(0,T;L^2(\mathbb{R}^3)))^9.
\end{aligned} \tag{2.19}$$

On the other hand, rewrite (2.1b) as

$$-\triangle\mathbf{u} - \nabla(\nabla\cdot\mathbf{u}) = -\rho\mathbf{u}_t - \rho\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}} - \nabla P + \mathbf{H}\cdot\nabla\mathbf{H} - \frac{1}{2}\nabla(|\mathbf{H}|^2) \equiv \mathbf{v}, \tag{2.20}$$

which is a system with a strongly elliptic left-hand side.

First, by a classical result on elliptic systems, there exists a positive constant C such that

$$\|\mathbf{u}\|_{2,q} \le C|\triangle\mathbf{u} + \nabla(\nabla\cdot\mathbf{u})|_q.$$

Then $\mathbf{v} \in (L^\infty(0,T;L^2(\mathbb{R}^3)))^3$ by (2.13) and (2.19), thus $\mathbf{u} \in (L^\infty(0,T;H^2(\mathbb{R}^3)))^3$. Second, since ρ is bounded from below and $\sqrt{\rho}\mathbf{u}_t \in (L^2(Q_T))^3$, we know that $\mathbf{u}_t \in (L^2(Q_T))^3$. Hence, by the Gagliardo–Nirenberg–Sobolev inequality, we get

$$\|\mathbf{u}_t\|_{L^2_T(L^q(\mathbb{R}^3))} \le C\|\mathbf{u}_t\|_{L^2(Q_T)}^\theta\|\nabla\mathbf{u}_t\|_{L^2(Q_T)}^{1-\theta} \quad \text{as } q\in(3,6],$$

for some $\theta \in [0,1)$. This implies that, by (2.19), $\mathbf{u}_t \in (L^2(0,T;L^q(\mathbb{R}^3)))^3$, then $\mathbf{v} \in (L^2(0,T;L^q(\mathbb{R}^3)))^3$. A similar consideration leads to $\mathbf{u} \in (L^2(0,T;W^{2,q}(\mathbb{R}^3)))^3$. Hence, we can conclude that $\mathbf{u} \in \Psi$.

2.4. *Existence for* (1.1)

The above argument guarantees the existence and uniqueness of the solution to the system (2.1) which enable us to define the map $\mathbf{u} = G(\bar{\mathbf{u}})$ given by the composition of

$$f : \bar{\mathbf{u}} \to \rho(\bar{\mathbf{u}}), \quad g : \bar{\mathbf{u}} \to \mathbf{H}(\bar{\mathbf{u}}), \quad d : (\rho(\bar{\mathbf{u}}), \bar{\mathbf{u}}, \mathbf{H}(\bar{\mathbf{u}})) \to \mathbf{u}.$$

Obviously, the fixed point of $G : \Psi \to \Psi$ is the solution of the system (1.1). To find a fixed point of G, we will use the Schauder–Tychonoff fixed-point theorem ([31, Theorem 5.28]).

Consider the set

$$\begin{aligned} M = \{\phi(x,t) \in \mathbb{R}^3 \mid \max(&\|\phi\|_{L^2_T(W^{2,q}(\mathbb{R}^3)\cap H^2(\mathbb{R}^3))}, \|\sqrt{\rho}\phi_t\|_{L^\infty_T(L^2(\mathbb{R}^3))}, \\ &\|\phi\|_{L^\infty_T(H^2(\mathbb{R}^3))}, \|\phi_t\|_{L^2_T(H^1(\mathbb{R}^3))}) \le r\} \end{aligned}$$

with

$$r^2 = \sigma(|\triangle \mathbf{u}_0|_2^2 + |\mathbf{u}_0|_\infty^2 |\nabla \mathbf{u}_0|_2^2 + \|\rho_0\|^2_{W^{1,q}\cap H^1} + \|\mathbf{H}_0\|^2_{W^{1,q}\cap H^1})/\alpha$$

for some suitable sufficiently large constant $\sigma \ge 1$, where $q \in (3,6]$. Clearly, M is a compact set in $(L^2(Q_T))^3$. As we are going to use a fixed-point theorem, we need to show that $G(M) \subset M$ and G is continuous in M with respect to the norm in $(L^2(Q_T))^3$.

We first prove that $G(M) \subset M$ for some $T = \overline{T}$. Indeed, assuming $\bar{\mathbf{u}} \in M$, from (2.3), (2.6), and (2.11), by Hölder's inequality, we know that for $3 < q \le 6$,

$$\begin{aligned} \alpha \exp(-Cr\sqrt{t}) \le \rho(x,t) &\le \exp(Cr\sqrt{t}), \\ |\nabla\rho|_q &\le (|\nabla\rho_0|_q + \exp(Cr\sqrt{t})r\sqrt{t})\exp(Cr\sqrt{t}), \\ |\nabla\mathbf{H}|_q &\le (|\nabla\mathbf{H}_0|_q + Cr\sqrt{t})\exp(Cr\sqrt{t}). \end{aligned} \tag{2.21}$$

Hence, from (2.18) and (2.21), it follows that

$$\begin{aligned} &\frac{1}{2}|\sqrt{\rho}\mathbf{u}_t|_2^2 + \frac{1}{2}\int_0^t |\nabla\mathbf{u}_t|_2^2 d\tau \\ &\quad \le \frac{1}{2}|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2^2 + C_\delta t(r^2+\delta)\|\sqrt{\rho}\mathbf{u}_t\|^2_{L^\infty_t(L^2(\mathbb{R}^3))} + C_\delta t(r^4+r^6) + Ctr^2. \end{aligned} \tag{2.22}$$

Since if we multiply (2.1b) by $\mathbf{u}_t$, integrate over $\mathbb{R}^3$ and let $t = 0$, then

$$\begin{aligned}|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2^2 &= \int_{\mathbb{R}^3}\bigg(\triangle\mathbf{u}_0 + \nabla(\nabla\cdot\mathbf{u}_0) - \rho_0\mathbf{u}_0\cdot\nabla\mathbf{u}_0 - \nabla P(\rho_0)\\ &\qquad + \mathbf{H}_0\cdot\nabla\mathbf{H}_0 - \frac{1}{2}\nabla(|\mathbf{H}_0|^2)\bigg)\cdot\mathbf{u}_t(0)\,dx\\ &\le \frac{1}{\sqrt{\alpha}}(|\triangle\mathbf{u}_0|_2 + |\nabla(\nabla\cdot\mathbf{u}_0)|_2 + |\rho_0|_\infty|\mathbf{u}_0|_\infty|\nabla\mathbf{u}_0|_2\\ &\qquad + \|P'\|_{C^0_{\rm loc}}|\nabla\rho_0|_2 + 2|\mathbf{H}_0|_\infty|\nabla\mathbf{H}_0|_2)|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2,\end{aligned}$$

i.e.

$$\begin{aligned}|\sqrt{\rho(0)}\mathbf{u}_t(0)|_2 \le \frac{1}{\sqrt{\alpha}}(|\triangle\mathbf{u}_0|_2 + |\nabla(\nabla\cdot\mathbf{u}_0)|_2 + |\rho_0|_\infty|\mathbf{u}_0|_\infty|\nabla\mathbf{u}_0|_2\\ + \|P'\|_{C^0_{\rm loc}}|\nabla\rho_0|_2 + 2|\mathbf{H}_0|_\infty|\nabla\mathbf{H}_0|_2).\end{aligned}\tag{2.23}$$

Combining (2.22) and (2.23), taking δ and $\overline{T}$ suitably small, we derive that

$$\begin{aligned}\|\sqrt{\rho}\mathbf{u}_t\|^2_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))} + \|\nabla\mathbf{u}_t\|^2_{L^2(Q_{\overline{T}})} \le \frac{1}{3}r^2,\\ \|\sqrt{\rho}\mathbf{u}_t\|^2_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))} + \|\mathbf{u}_t\|^2_{L^2_{\overline{T}}(H^1(\mathbb{R}^3))} \le \frac{2}{3}r^2.\end{aligned}\tag{2.24}$$

Then, we estimate the norm $\|\mathbf{u}\|_{L^\infty_{\overline{T}}(H^2(\mathbb{R}^3))}$ and the norm $\|\mathbf{u}\|_{L^2_{\overline{T}}(W^{2,q}(\mathbb{R}^3))}$. Indeed, taking advantage of (2.20), on the one hand, we get

$$\begin{aligned}&\|\triangle\mathbf{u} + \nabla(\nabla\cdot\mathbf{u})\|_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))}\\ &\quad\le \|\sqrt{\rho}\|_{L^\infty(Q_{\overline{T}})}\|\sqrt{\rho}\mathbf{u}_t\|_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))} + \|\rho\|_{L^\infty(Q_{\overline{T}})}\|\bar{\mathbf{u}}\|_{L^\infty(Q_{\overline{T}})}\|\nabla\bar{\mathbf{u}}\|_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))}\\ &\qquad + C\|\nabla\rho\|_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))} + C\|\nabla\mathbf{H}\|_{L^\infty_{\overline{T}}(L^2(\mathbb{R}^3))},\end{aligned}$$

which leads to

$$\|\mathbf{u}\|^2_{L^\infty_{\overline{T}}(H^2(\mathbb{R}^3))} \le r^2$$

from a classical result for elliptic systems. On the other hand, we have, also by the classical result on elliptic systems,

$$\begin{aligned}\int_0^{\overline{T}}\|\mathbf{u}\|^2_{2,q}\,dt &\le C\int_0^{\overline{T}}(|\bar{\mathbf{u}}|^2_{L^\infty}|\nabla\bar{\mathbf{u}}|^2_q + |\nabla\rho|^2_q + |\mathbf{u}_t|^2_q + |\mathbf{H}|^2_\infty|\nabla\mathbf{H}|^2_q)\,dt\\ &\le Cr^4\overline{T} + r^2\overline{T}\\ &\le r^2\end{aligned}$$

for some suitable small $\overline{T}$. Hence, we have shown that $G(M) \subset M$.

Next, we prove the continuity of G in M. We observe that if $\{\bar{\mathbf{u}}_n\}_{n=1}^{\infty} \subset M$, then there exists a subsequence (still denoted by $\{\bar{\mathbf{u}}_n\}_{n=1}^{\infty}$) such that

$$\bar{\mathbf{u}}_n \to \bar{\mathbf{u}} \quad \text{strongly in } M \text{ as } n \to \infty.$$

Let ρ_n and ρ be the solutions of

$$\rho_{nt} + \nabla \cdot (\rho_n \bar{\mathbf{u}}_n) = 0, \quad \rho_n(0) = \rho_0,$$

and

$$\rho_t + \nabla \cdot (\rho \bar{\mathbf{u}}) = 0, \quad \rho(0) = \rho_0,$$

respectively. Define $\overline{\rho}_n = \rho_n - \rho$, then $\overline{\rho}_n$ satisfies

$$\overline{\rho}_{nt} + \bar{\mathbf{u}}_n \cdot \nabla \overline{\rho}_n + (\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) \cdot \nabla \rho + \overline{\rho}_n \nabla \cdot \bar{\mathbf{u}}_n + \rho \nabla \cdot (\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) = 0, \quad \overline{\rho}_n(0) = 0. \tag{2.25}$$

Multiplying (2.25) by $\overline{\rho}_n$, integrating over Q_T, and applying Gronwall's inequality, it is easy to derive that

$$|\overline{\rho}_n|_2^2 \le \exp(Cr\overline{T}) \int_0^{\overline{T}} (|(\bar{\mathbf{u}} - \bar{\mathbf{u}}_n) \cdot \nabla \rho|_2^2 + |\rho \nabla \cdot (\bar{\mathbf{u}}_n - \bar{\mathbf{u}})|_2^2)\, dt,$$

which implies that $\rho_n \to \rho$ strongly in $L^\infty(0, \overline{T}; L^2(\mathbb{R}^3))$.

Similarly, we can show that $\mathbf{H}_n \to \mathbf{H}$ strongly in $(L^\infty(0, \overline{T}; L^2(\mathbb{R}^3)))^3$.

Now let $\mathbf{u}_n$ and $\mathbf{u}$ be the solutions of (2.1b) corresponding to $\bar{\mathbf{u}}_n$ and $\bar{\mathbf{u}}$ with

$$\mathbf{u}_n(0) = \mathbf{u}(0) = \mathbf{u}_0.$$

Then we have, defining $\mathbf{U}_n = \mathbf{u}_n - \mathbf{u}$ and $\bar{\mathbf{U}}_n = \bar{\mathbf{u}}_n - \bar{\mathbf{u}}$,

$$\begin{aligned} &\rho_n \mathbf{U}_{nt} - \triangle \mathbf{U}_n - \nabla \nabla \cdot \mathbf{U}_n \\ &\quad = -\overline{\rho}_n \mathbf{u}_t - \rho_n \bar{\mathbf{U}}_n \cdot \nabla \bar{\mathbf{u}}_n - \overline{\rho}_n \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}_n - \rho \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{U}}_n \\ &\qquad + \mathbf{H}_n \cdot \nabla \mathbf{H}_n - \mathbf{H} \cdot \nabla \mathbf{H} - \nabla P(\rho_n) + \nabla P(\rho). \end{aligned} \tag{2.26}$$

Multiplying (2.26) by $\mathbf{U}_{nt}$, integrating over $Q_{\overline{T}}$, and take advantage of the convergence of ρ_n and $\mathbf{H}_n$, we can prove as a routine matter that

$$\nabla \bar{\mathbf{U}}_n \to 0 \quad \text{strongly in } (L^2(Q_{\overline{T}}))^9$$

and

$$\sqrt{\rho_n} \mathbf{U}_{nt} \to 0 \quad \text{strongly in } (L^2(Q_{\overline{T}}))^3.$$

Due to the convergence of ρ_n, we deduce that

$$\mathbf{U}_{nt} \to 0 \quad \text{strongly in } (L^2(Q_{\overline{T}}))^3.$$

Hence, by using the identity $\mathbf{U}_n(t) = \int_0^t \mathbf{U}_{nt}\, d\tau$ $(\mathbf{U}_n(0) = 0)$, we get

$$\mathbf{U}_n \to 0 \quad \text{strongly in } (L^2(Q_{\overline{T}}))^3.$$

Thus, the map G is continuous in M. The existence of a local solution is completely proved.

3. Uniqueness

We proceed to prove the uniqueness of the solution by the same procedure as that used for the continuity of G. We have already proved that for $3 < q \leq 6$,

$$\begin{aligned} &\mathbf{u}_t \in (L^2(0,\overline{T}; L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)))^3, \\ &\nabla\rho \in (L^2(0,\overline{T}; L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)))^3, \\ &\nabla\mathbf{H} \in (L^2(0,\overline{T}; L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)))^9. \end{aligned}$$

Using the standard interpolation, we get

$$\begin{aligned} &\mathbf{u}_t \in (L^2(0,\overline{T}; L^3(\mathbb{R}^3)))^3, \\ &\nabla\rho \in (L^2(0,\overline{T}; L^3(\mathbb{R}^3)))^3, \\ &\nabla\mathbf{H} \in (L^2(0,\overline{T}; L^3(\mathbb{R}^3)))^9, \end{aligned}$$

where

$$\frac{1}{3} = \frac{\theta}{2} + \frac{1-\theta}{q}.$$

Now, assume that $\mathbf{u}_1$, $\mathbf{u}_2$ satisfy (1.1) for some $T > 0$ and define

$$\rho := \rho(\mathbf{u}_1) - \rho(\mathbf{u}_2) = \rho_1 - \rho_2, \quad \mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2, \quad \mathbf{H} := \mathbf{H}(\mathbf{u}_1) - \mathbf{H}(\mathbf{u}_2) = \mathbf{H}_1 - \mathbf{H}_2.$$

Then, we have

$$\rho_t + \nabla\rho\cdot\mathbf{u}_1 + \nabla\rho_2\cdot\mathbf{u} + \rho\nabla\cdot\mathbf{u}_1 + \rho_2\nabla\cdot\mathbf{u} = 0, \tag{3.1}$$

with $\rho(0) = 0$. Multiplying (3.1) by ρ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\rho|_2^2 - \frac{1}{2}\int_{\mathbb{R}^3}|\rho|^2\nabla\cdot\mathbf{u}_1\,dx + \int_{\mathbb{R}^3}\rho\nabla\rho_2\cdot\mathbf{u}\,dx + \int_{\mathbb{R}^3}|\rho|^2\nabla\cdot\mathbf{u}_1\,dx \\ &\quad + \int_{\mathbb{R}^3}\rho\rho_2\nabla\cdot\mathbf{u}\,dx = 0. \end{aligned}$$

Combining the Cauchy–Schwarz inequality, the Hölder inequality, and $|\mathbf{u}|_6 \leq C|\nabla\mathbf{u}|_2$, we get

$$\begin{aligned} \frac{d}{dt}|\rho|_2^2 &\leq |\nabla\cdot\mathbf{u}_1|_\infty|\rho|_2^2 + 2|\mathbf{u}|_6|\rho\nabla\rho_2|_{\frac{6}{5}} + \varepsilon|\nabla\mathbf{u}|_2^2 + C_\varepsilon|\rho_2|_\infty^2|\rho|_2^2 \\ &\leq |\nabla\cdot\mathbf{u}_1|_\infty|\rho|_2^2 + \varepsilon|\nabla\mathbf{u}|_2^2 + C_\varepsilon|\rho\nabla\rho_2|_{\frac{6}{5}}^2 + \varepsilon|\nabla\mathbf{u}|_2^2 + C_\varepsilon|\rho_2|_\infty^2|\rho|_2^2 \\ &\leq |\nabla\cdot\mathbf{u}_1|_\infty|\rho|_2^2 + \varepsilon|\nabla\mathbf{u}|_2^2 + C_\varepsilon|\nabla\rho_2|_3^2|\rho|_2^2 + \varepsilon|\nabla\mathbf{u}|_2^2 + C_\varepsilon|\rho_2|_\infty^2|\rho|_2^2 \\ &\leq \eta_1(\varepsilon)|\rho|_2^2 + 2\varepsilon|\nabla\mathbf{u}|_2^2, \end{aligned} \tag{3.2}$$

where $\eta_1(\varepsilon) = |\nabla\cdot\mathbf{u}_1|_\infty + C_\varepsilon(|\nabla\rho_2|_3^2 + |\rho_2|_\infty^2),\ \varepsilon > 0$.

Similarly, we have

$$\mathbf{H}_t + \mathbf{u}_1 \cdot \nabla \mathbf{H} + \mathbf{u} \cdot \nabla \mathbf{H}_2 = (\nabla \mathbf{u}_1 - (\nabla \cdot \mathbf{u}_1)\mathbf{I})\mathbf{H} + (\nabla \mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{I})\mathbf{H}_2, \quad (3.3)$$

with $\mathbf{H}(0) = 0$. Using the same technique as for ρ, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\mathbf{H}|_2^2 - \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{H}|^2 \nabla \cdot \mathbf{u}_1 \, dx + \int_{\mathbb{R}^3} \mathbf{H} \cdot (\mathbf{u} \cdot \nabla \mathbf{H}_2) \, dx \\
&\quad = \int_{\mathbb{R}^3} \mathbf{H} \cdot (\nabla \mathbf{u}_1 - (\nabla \cdot \mathbf{u}_1)\mathbf{I})\mathbf{H} \, dx + \int_{\mathbb{R}^3} \mathbf{H} \cdot (\nabla \mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{I})\mathbf{H}_2 \, dx,
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}|\mathbf{H}|_2^2 &\le C|\nabla \cdot \mathbf{u}_1|_\infty |\mathbf{H}|_2^2 + \varepsilon|\nabla \mathbf{u}|_2^2 + C_\varepsilon |\mathbf{H}_2|_\infty^2 |\mathbf{H}|_2^2 + \varepsilon|\nabla \mathbf{u}|_2^2 + C_\varepsilon |\mathbf{H} \cdot \nabla \mathbf{H}_2|_{\frac{6}{5}}^2 \\
&\le C|\nabla \cdot \mathbf{u}_1|_\infty |\mathbf{H}|_2^2 + 2\varepsilon|\nabla \mathbf{u}|_2^2 + C_\varepsilon |\mathbf{H}_2|_\infty^2 |\mathbf{H}|_2^2 + C_\varepsilon |\nabla \mathbf{H}_2|_3^2 |\mathbf{H}|_2^2 \\
&\le \eta_2(\varepsilon)|\mathbf{H}|_2^2 + 2\varepsilon|\nabla \mathbf{u}|_2^2, \qquad (3.4)
\end{aligned}$$

where $\eta_2(\varepsilon) = C|\nabla \cdot \mathbf{u}_1|_\infty + C_\varepsilon(|\mathbf{H}_2|_\infty^2 + |\nabla \mathbf{H}_2|_3^2), \ \varepsilon > 0$.

For $\mathbf{u}_i$, $i = 1, 2$,

$$\begin{aligned}
\rho_i \mathbf{u}_{it} - \triangle \mathbf{u}_i - \nabla \cdot \mathbf{u}_i &= -\rho_i \mathbf{u}_i \cdot \nabla \mathbf{u}_i - \nabla P(\rho_i) + \mathbf{H}_i \cdot \nabla \mathbf{H}_i - \frac{1}{2}\nabla(|\mathbf{H}_i|^2), \\
\mathbf{u}_i(0) &= \mathbf{u}_0.
\end{aligned}$$

It is easy to derive

$$\begin{aligned}
&\rho_1 \mathbf{u}_t - \triangle \mathbf{u} - \nabla(\nabla \cdot \mathbf{u}) \\
&\quad = -\rho \mathbf{u}_{2t} - \rho_1 \mathbf{u} \cdot \nabla \mathbf{u}_1 - \rho \mathbf{u}_2 \cdot \nabla \mathbf{u}_1 - \rho_2 \mathbf{u}_2 \cdot \nabla \mathbf{u} - \nabla P(\rho_1) + \nabla P(\rho_2) \\
&\qquad + \mathbf{H}_1 \cdot \nabla \mathbf{H}_1 - \mathbf{H}_2 \cdot \nabla \mathbf{H}_2 - \frac{1}{2}\nabla(|\mathbf{H}_1|^2) + \frac{1}{2}\nabla(|\mathbf{H}_2|^2), \qquad (3.5)
\end{aligned}$$

with $\mathbf{u}(0) = 0$.

Using once more the same technique as for $\rho, \mathbf{H}$, and bearing in mind the continuity equation, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\sqrt{\rho}_1 \mathbf{u}|_2^2 + |\nabla \mathbf{u}|_2^2 + |\nabla \cdot \mathbf{u}|_2^2 \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}|^2 \nabla \cdot (\rho_1 \mathbf{u}_1) \, dx - \int_{\mathbb{R}^3} \Big(\rho \mathbf{u}_{2t} \cdot \mathbf{u} + \rho_1 \mathbf{u} \cdot \nabla \mathbf{u}_1 \cdot \mathbf{u} + \rho \mathbf{u}_2 \cdot \nabla \mathbf{u}_1 \cdot \mathbf{u} \\
&\qquad + \rho_2 \mathbf{u}_2 \cdot \nabla \mathbf{u} \cdot \mathbf{u} + \nabla P(\rho_1) \cdot \mathbf{u} - \nabla P(\rho_2) \cdot \mathbf{u} - \mathbf{H} \cdot \nabla \mathbf{H}_1 \cdot \mathbf{u} - \mathbf{H}_2 \cdot \nabla \mathbf{H} \cdot \mathbf{u} \\
&\qquad + \frac{1}{2}(\nabla(|\mathbf{H}_1|^2) - \nabla(|\mathbf{H}_2|^2)) \cdot \mathbf{u} \Big) \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \mathbf{u} \cdot (\rho_1 \mathbf{u}_1)\, dx + \int_{\mathbb{R}^3} (\rho \mathbf{u}_{2t} \cdot \mathbf{u} - \rho_1 \mathbf{u} \cdot \nabla \mathbf{u}_1 \cdot \mathbf{u} - \rho \mathbf{u}_2 \cdot \nabla \mathbf{u}_1 \cdot \mathbf{u} \\
&\quad - \rho_2 \mathbf{u}_2 \cdot \nabla \mathbf{u} \cdot \mathbf{u} - \nabla P(\rho_1) \cdot \mathbf{u} + \nabla P(\rho_2) \cdot \mathbf{u} + \mathbf{H} \cdot \nabla \mathbf{H}_1 \cdot \mathbf{u} + \mathbf{H}_2 \cdot \nabla \mathbf{H} \cdot \mathbf{u} \\
&\quad - \nabla \mathbf{H}_1 \cdot \mathbf{H} \cdot \mathbf{u} - \nabla \mathbf{H} \cdot \mathbf{H}_2 \cdot \mathbf{u})\, dx \\
&\le C_\varepsilon |\rho_1|_\infty^2 |\mathbf{u}_1|_\infty^2 |\mathbf{u}|_2^2 + \varepsilon |\nabla \mathbf{u}|_2^2 + \varepsilon |\nabla \mathbf{u}|_2^2 + C_\varepsilon |\mathbf{u}_{2t}|_3^2 |\rho|_2^2 + |\sqrt{\rho}_1|_\infty |\nabla \mathbf{u}_1|_\infty |\mathbf{u}|_2^2 \\
&\quad + |\mathbf{u}_2|_\infty |\nabla \mathbf{u}_1|_\infty (|\rho|_2^2 + |\mathbf{u}|_2^2) + C_\varepsilon |\rho_2|_\infty^2 |\mathbf{u}_2|_\infty^2 |\mathbf{u}|_2^2 + \varepsilon |\nabla \mathbf{u}|_2^2 \\
&\quad + C_\varepsilon \|P'\|_{C^0_{\mathrm{loc}}}^2 |\rho|_2^2 + \varepsilon |\nabla \mathbf{u}|_2^2 + \varepsilon |\nabla \mathbf{u}|_2^2 + C_\varepsilon |\nabla \mathbf{H}_1|_2^2 |\mathbf{H}|_2^2 \\
&\quad + C_\varepsilon (|\mathbf{H}_1|_\infty^2 + |\mathbf{H}_2|_\infty^2) |\mathbf{H}|_2^2 + \varepsilon |\nabla \cdot \mathbf{u}|_2^2 \\
&\le 5\varepsilon |\nabla \mathbf{u}|_2^2 + \varepsilon |\nabla \cdot \mathbf{u}|_2^2 + \eta_3(\varepsilon)(|\rho|_2^2 + |\mathbf{u}|_2^2 + |\mathbf{H}|_2^2), \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_3(\varepsilon) = C_\varepsilon (&|\rho_1|_\infty^2 |\mathbf{u}_1|_\infty^2 + |\mathbf{u}_{2t}|_3^2 + |\sqrt{\rho}_1|_\infty |\nabla \mathbf{u}_1|_\infty + |\mathbf{u}_2|_\infty |\nabla \mathbf{u}_1|_\infty + |\rho_2|_\infty^2 |\mathbf{u}_2|_\infty^2 \\
&+ \|P'\|_{C^0_{\mathrm{loc}}}^2 + |\nabla \mathbf{H}_1|_2^2 + |\mathbf{H}_1|_\infty^2 + |\mathbf{H}_2|_\infty^2).
\end{aligned}
$$

Summing up (3.2), (3.4) and (3.6), by taking $\varepsilon \le \frac{1}{14}$, we obtain

$$
\begin{aligned}
\frac{d}{dt}(|\sqrt{\rho}_1 \mathbf{u}|_2^2 + |\rho|_2^2 + |\mathbf{H}|_2^2) + |\nabla \mathbf{u}|_2^2 &\le 2(\eta_1(\varepsilon) + \eta_2(\varepsilon) + \eta_3(\varepsilon))(|\mathbf{u}|_2^2 + |\rho|_2^2 + |\mathbf{H}|_2^2) \\
&\le \eta(\varepsilon, t)(|\sqrt{\rho}_1 \mathbf{u}|_2^2 + |\rho|_2^2 + |\mathbf{H}|_2^2), \qquad (3.7)
\end{aligned}
$$

where

$$
\eta(\varepsilon, t) = 2(\eta_1(\varepsilon) + \eta_2(\varepsilon) + \eta_3(\varepsilon)) \max\left\{1, \frac{1}{\alpha} \exp(Cr\sqrt{t})\right\}.
$$

And, the integrability of $\eta(\varepsilon, t)$ with respect to t on $[0, T]$ comes from the regularity of $\mathbf{u}_1, \mathbf{u}_2$ and the estimates in Lemmas 2.1 and 2.2 for ρ_i, $\mathbf{H}_i$ with $i = 1, 2$. Hence,

$$
|\sqrt{\rho}_1 \mathbf{u}|_2^2 + |\rho|_2^2 + |\mathbf{H}|_2^2 = 0 \quad \text{for all } t \in (0, T)
$$

follows from Gronwall's inequality, and consequently,

$$
\mathbf{u} \equiv 0, \quad \rho \equiv 0, \quad \mathbf{H} \equiv 0 \quad \text{on } Q_T.
$$

The proof of uniqueness is complete.

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