

# Formation of singularities of solutions to the three-dimensional Euler–Boltzmann equations in radiation hydrodynamics

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## Abstract

The Cauchy problem for the three-dimensional Euler–Boltzmann equations of a polytropic, ideal and isentropic fluid in radiation hydrodynamics is considered. The formation of singularities in smooth solutions is studied. It is proved that some  $C^1$  solutions, regardless of the size of the initial disturbance, will develop singularities in a finite time provided that the initial disturbance satisfies certain conditions.

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## 1. Introduction

In this paper we are interested in the finite-time formation of singularities in three-dimensional radiation hydrodynamics. Radiation hydrodynamics (cf [13, 14]) is concerned with the propagation of thermal radiation and the effect of this radiation on the hydrodynamics describing the fluid motion. The importance of thermal radiation in physical problems increases as the temperature is raised. At moderate temperatures, the role of the radiation is primarily transporting energy by radiative processes. At higher temperatures (say, millions of kelvins), the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities. In this case, the radiation field significantly affects the dynamics of the fluid. The hydrodynamics with explicit account of radiation energy and momentum contributions constitutes the character of radiation hydrodynamics. The theory of radiation hydrodynamics finds a wide range of applications, including diverse astrophysical phenomena such as waves and oscillations in stellar atmospheres and envelopes, nonlinear

stellar pulsation, supernova explosions, stellar winds and many other areas. The mathematical equations governing the radiation hydrodynamics are the Euler equations of compressible fluids coupled with the Boltzmann equation of particle transport. It is very important to understand the solutions of the Euler–Boltzmann equations in order to obtain insights into the radiation hydrodynamics and related physical phenomena as well as applications. However, solving the Euler–Boltzmann equations is challenging because of the complexity and mathematical difficulty.

We use  $I(x, t, \nu, \Omega)$  to denote the specific intensity of radiation (at time  $t$ ) at space point  $x \in \mathbb{R}^3$ , with frequency  $\nu$  in a direction  $\Omega \in S^2$  (the unit sphere of  $\mathbb{R}^3$ ); then the system of partial differential equations of three-dimensional isentropic radiation hydrodynamics (cf [13, 14]) consists of the following Boltzmann equation:

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\nu, \Omega)}{\partial t} + \Omega \cdot \nabla I(\nu, \Omega) &= S(\nu, \rho) - \sigma_a(\nu, \rho) I(\nu, \Omega) \\ &+ \int_0^\infty d\nu' \int_{S^2} \left( \frac{\nu}{\nu'} \sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega, \rho) I(\nu', \Omega') - \sigma_s(\nu \rightarrow \nu', \Omega \cdot \Omega', \rho) I(\nu, \Omega) \right) d\Omega', \end{aligned} \quad (1.1)$$

and the Euler equations

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0, \\ \left( \rho u + \frac{1}{c^2} F_r \right)_t + \nabla p + \nabla \cdot (\rho u \otimes u + P_r) &= 0, \end{aligned} \quad (1.2)$$

where  $c$  denotes the light speed,  $S(\nu, \rho) = S(x, t, \nu, \Omega, \rho)$  denotes the rate of energy emission due to spontaneous processes,  $\sigma_a(\nu, \rho) = \sigma_a(x, t, \nu, \Omega, \rho)$  denotes the absorption coefficient. Similar to absorption, a photon can undergo scattering interactions with matter, and the scattering interaction serves to change the photon's characteristics  $\nu'$  and  $\Omega'$  to a new set of characteristics  $\nu$  and  $\Omega$ ; this leads to the definition of the 'differential scattering coefficient'  $\sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega, \rho) = \sigma_s(x, t, \nu' \rightarrow \nu, \Omega' \cdot \Omega, \rho)$ . In the Euler equations (1.2),  $\rho = \rho(x, t)$  is the density,  $u = u(x, t) \in \mathbb{R}^3$  is the velocity,  $p = p(\rho)$  is the pressure,  $F_r$  and  $P_r$  represent the radiative flux and the radiative pressure tensor respectively defined by

$$\begin{aligned} F_r &= \int_0^\infty d\nu \int_{S^2} \Omega I(\nu, \Omega) d\Omega, \\ P_r &= \frac{1}{c} \int_0^\infty d\nu \int_{S^2} \Omega \otimes \Omega I(\nu, \Omega) d\Omega. \end{aligned} \quad (1.3)$$

In this paper we consider only polytropic ideal gases, namely

$$p = A e^{\bar{S}} \rho^\gamma, \quad (1.4)$$

with  $A > 0$  and entropy  $S \equiv \bar{S}$  constant,  $\gamma > 1$  being the adiabatic index.

There have been many results on the formation of singularities for the compressible Euler equations. For the one-dimensional Euler equations, one can use the method of characteristics (cf [1, 3, 4, 7, 8, 10, 11]). For systems with multi-dimensional space variables, the method of characteristics has not been proved tractable. An efficient method, involving the use of averaged quantities, was developed in [16] for hyperbolic systems of conservation laws and was further refined in [17] for the three-dimensional Euler equations. See also [15] for the two-dimensional Euler equation and [12, 18] for the multi-dimensional systems of conservation laws. But there are few mathematical results on the general radiation hydrodynamical system (1.1)–(1.4) because of their complex structure. Recently, Jiang–Zhong in [6] obtained the local existence of  $C^1$  solutions for the Cauchy problems of the general radiation hydrodynamical system, and

showed the finite-time blowup of  $C^1$  solutions under the assumption that the initial data are large, in particular, the initial flow velocity must be supersonic in some region (cf [6, 17]). In this paper we study the finite-time formation of singularities for the Euler–Boltzmann system and prove that some  $C^1$  solutions to the equations of radiation hydrodynamics cannot exist for all time regardless of the size of the initial disturbance. This paper is motivated by [17]. We will closely follow [17] to develop the proof in the context of radiation hydrodynamics. In addition, we need to overcome the complexity from the radiation governed by the Boltzmann equation (1.1), which requires some new ideas and new ingredients in the proof.

The rest of this paper is organized as follows. In section 2, we reformulate the problem and state the main result. In section 3, we prove the main result of this paper.

## 2. Reformulation and main result

Specifically, we consider the manifestation in the equation of transfer of the quantum statistics (i.e. (1.1)) obeyed by photons. Since photons are bosons, both the processes of emissions and scattering are enhanced by the number of photons already in the final state following the interaction. This enhancement is generally referred to as resulting from ‘induced processes’ (see, e.g., [14]). The quantitative statement of this enhancement is simply stated as follows: if  $Z$  represents the basic probability of a photon event (emission or scattering, i.e.  $S(\nu, \rho)$  or  $\sigma_s(\nu, \rho)$ ) then, due to induced effects, the actual probability  $Z'$  is given by (see, e.g., [5])

$$Z' = Z(1 + n),$$

where  $n$  is the number of photons in the final state of the transition. In the ‘induced processes’ case (see, e.g., [9]),

$$n = \frac{c^2}{2h\nu^3} I(\nu, \Omega),$$

and thus

$$Z' = Z \left( 1 + \frac{c^2 I}{2h\nu^3} \right),$$

where  $h$  is the Planck constant.

Another item of interest to consider here is the concept of local thermodynamics equilibrium (LTE) (see, e.g., [2, 13, 14]). To see the effect of the LTE assumption on equation (1.1), it is convenient to eliminate  $S, \sigma_a$  in (1.1) in favour of  $\bar{B}$  and  $a(\nu, \rho)$  defined by the relationships

$$S(\nu, \rho) = a(\nu, \rho) \bar{B}(\nu), \quad \sigma_a(\nu, \rho) = a(\nu, \rho) \left( 1 + \frac{c^2 \bar{B}(\nu)}{2h\nu^3} \right),$$

and assume that  $\sigma_s = 0$ , where  $\bar{B}$  is a function of  $\nu$  only, and the absorption coefficient  $a(\nu, \rho) = a(x, t, \nu, \Omega, \rho)$ , we assume throughout this paper that  $a(\nu, \rho) > 0$ ; more comments on  $\bar{B}$  and  $a(\nu, \rho)$  can be found in remarks 2.2 and 3.1 of [6] as well as in [14]. Thus, from the ‘induced processes’ and the LTE assumption together,  $S(\nu, \rho), \sigma_a(\nu, \rho)$  in (1.1) can be written as

$$\begin{aligned} S(\nu, \rho) &= a(\nu, \rho) \bar{B}(\nu) \left( 1 + \frac{c^2 I(\nu, \Omega)}{2h\nu^3} \right), \\ \sigma_a(\nu, \rho) &= a(\nu, \rho) \left( 1 + \frac{c^2 \bar{B}(\nu)}{2h\nu^3} \right). \end{aligned} \tag{2.1}$$

Using (2.1), we can rewrite equations (1.1) and (1.2) as

$$\frac{1}{c} \frac{\partial I(v, \Omega)}{\partial t} + \Omega \cdot \nabla I(v, \Omega) = a(v, \rho)(\bar{B}(v) - I), \quad (2.2)$$

$$\rho_t + \nabla \cdot (\rho u) = 0,$$

$$(\rho u)_t + \nabla p + \nabla \cdot (\rho u \otimes u) = -\frac{1}{c} \int_0^\infty dv \int_{S^2} \Omega (a(v, \rho)(\bar{B}(v) - I)) d\Omega. \quad (2.3)$$

We consider the Cauchy problem of (2.2) and (2.3) with the following initial data:

$$\begin{aligned} I(x, 0, v, \Omega) &= I^0(x, v, \Omega), & I^0(x, v, \Omega) &= \bar{B}(v), & |x| &\geq R, \\ \rho(x, 0) &= \rho^0(x) > 0, & \rho^0(x) &= \bar{\rho}, & |x| &\geq R, \\ u(x, 0) &= u^0(x), & u^0(x) &= \bar{u} (= 0), & |x| &\geq R, \end{aligned} \quad (2.4)$$

where  $R > 0$ ,  $\bar{\rho} > 0$  are some constant.

Let  $(\rho, u, I)$  be the local  $C^1$  solution to the Cauchy problem (2.2)–(2.4) obtained in Jiang–Zhong [6].

**Lemma 2.1.**

- (1) If  $I^0(x, v, \Omega) \geq \bar{B}(v)$  for  $(x, v, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$ , then  $I(x, t, v, \Omega) \geq \bar{B}(v)$  for  $(x, v, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$  and  $t > 0$ .
- (2) If  $I^0(x, v, \Omega) = \bar{B}(v)$  for  $x \cdot \Omega < 0$  and  $v \in \mathbb{R}^+$ , then  $I(x, t, v, \Omega) = \bar{B}(v)$  for  $x \cdot \Omega < 0$ ,  $v \in \mathbb{R}^+$  and  $t > 0$ .

**Proof.** Since  $\bar{B}$  is independent of  $x$  and  $t$ , equation (2.2) can be rewritten as

$$\frac{1}{c} \frac{\partial (I - \bar{B})}{\partial t} + \Omega \cdot \nabla (I - \bar{B}) + a(v, \rho)(I - \bar{B}) = 0.$$

An application of the method of characteristics to the above equation shows that  $I(x, t, v, \Omega) - \bar{B}(v)$  has the same sign as  $I^0(x - c\Omega t, v, \Omega) - \bar{B}(v)$ .

- (1) If  $I^0(x, v, \Omega) \geq \bar{B}(v)$  for  $(x, v, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$ , then  $I^0(x - c\Omega t, v, \Omega) - \bar{B}(v) \geq 0$ , thus  $I(x, t, v, \Omega) - \bar{B}(v) \geq 0$ , i.e.  $I(x, t, v, \Omega) \geq \bar{B}(v)$ .
- (2) For  $x \cdot \Omega < 0$ ,  $I^0(x, v, \Omega) = \bar{B}(v)$ , then  $I^0(x - c\Omega t, v, \Omega) - \bar{B}(v) = 0$  since  $(x - c\Omega t) \cdot \Omega = x \cdot \Omega - ct|\Omega|^2 < 0$ , thus  $I(x, t, v, \Omega) - \bar{B}(v) = 0$ , i.e.  $I(x, t, v, \Omega) = \bar{B}(v)$ .  $\square$

The maximum speed of propagation of the front of a smooth disturbance is governed by the sound speed

$$\sigma = (\partial_\rho p(\bar{\rho}))^{1/2} = \left( A\gamma \bar{\rho}^{\gamma-1} e^{\bar{s}} \right)^{1/2}, \quad (2.5)$$

since  $\bar{u} = 0$ . More precisely, letting

$$D(t) = \{x \in \mathbb{R}^3 : |x| \geq R + \sigma t\},$$

we have the following property:

**Proposition 2.1 (Finite propagation speed).** *If  $(I, \rho, u)$  is a  $C^1$  solution of (2.2)–(2.4), then  $(I, \rho, u) \equiv (\bar{B}(v), \bar{\rho}, 0)$  on  $D(t)$ ,  $0 \leq t \leq T$  for any fixed  $T > 0$ .*

The proof of this proposition can be found in [6, proposition 3.1].

Now, we define the functions

$$m(t) = \int_{\mathbb{R}^3} (\rho(x, t) - \bar{\rho}) \, dx,$$

$$K(t) = \int_{\mathbb{R}^3} x \cdot \rho u \, dx + \frac{1}{c^2} \int_{\mathbb{R}^3} dx \int_0^\infty dv \int_{S^2} x \cdot \Omega (I - \bar{B}) \, d\Omega.$$

As in theorem 2.1 below, singularities in  $C^1$  solutions in three-dimensional radiation hydrodynamics are developed in a finite time if the initial data are large. The proof of theorem 2.1 is omitted since it is the same as that in [6, theorem 3.1] for general nonisentropic radiation hydrodynamical system.

**Theorem 2.1.** *Let  $(I, \rho, u)$  be a  $C^1$  solution of (2.2)–(2.4) for  $0 \leq t < T$ . If*

$$I^0(x, \nu, \Omega) \geq \bar{B}(\nu), \quad m(0) \geq 0, \tag{2.6}$$

$$K(0) \geq \frac{16\pi}{3} \sigma R^4 \left( \max_x \rho^0(x) + \frac{1}{c^3} \max_x \int_0^\infty dv \int_{S^2} (I^0 - \bar{B}) \, d\Omega \right), \tag{2.7}$$

then the life span  $T$  is finite.

**Remark 2.1.** To show one way in which (2.6) and (2.7) can be satisfied, we take the initial conditions as

$$\rho^0 \equiv \bar{\rho}, \quad I^0 \equiv \bar{B},$$

Then  $m(0) = 0$ , and (2.7) holds if

$$\int_{\mathbb{R}^3} x \cdot u^0(x) \, dx \geq \frac{16\pi}{3} \sigma R^4.$$

Comparing both sides, we find that the initial flow velocity must be supersonic in some region [17].

Our main result is theorem 2.2 below, which establishes the finite-time formation of singularities without any condition of largeness on the initial data such as (2.7). Let us define the functions

$$q^0(r) = \int_{|x|>r} |x|^{-1} (|x| - r)^2 (\rho^0(x) - \bar{\rho}) \, dx,$$

$$q^1(r) = \int_{|x|>r} |x|^{-3} (|x|^2 - r^2) x \cdot \rho^0(x) u^0(x) \, dx.$$

**Theorem 2.2 (Main result).** *Suppose that for some  $R_0$  with  $0 < R_0 < R$ ,*

$$q^0(r) > 0, \quad q^1(r) \geq 0, \tag{2.8}$$

for  $R_0 < r < R$ ; and for  $(x, \nu, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$ ,

$$I^0(x, \nu, \Omega) \geq \bar{B}(\nu), \quad \text{and in addition, } I^0(x, \nu, \Omega) = \bar{B}(\nu) \quad \text{if } x \cdot \Omega < 0. \tag{2.9}$$

Then the life span  $T$  of the  $C^1$  solution of (2.2)–(2.4) is finite.

**Remark 2.2.** Define

$$q^2(r, t) = \frac{1}{c} \int_{|x|>r} \int_0^\infty dv \int_{S^2} |x|^{-3} (|x|^2 - r^2) x \cdot \Omega a(\nu, \rho) (I - \bar{B}) \, d\Omega \, dx.$$

Condition (2.9) on the initial data of  $I$  and lemma 2.1 imply

$$q^2(r, t) \geq 0, \tag{2.10}$$

since  $I - \bar{B}$  is supported on  $x \cdot \Omega \geq 0$ .

In what follows, all generic constants will be denoted by  $C$  which may depend on the fixed constants  $R$  and  $R_0$ , but is independent of the initial data.

**3. Proof of the main result: theorem 2.2**

In this section, we prove our main result in theorem 2.2. For the sake of clarity, we begin with the case  $\gamma = 2$ , and later we will indicate what modifications are necessary for the general case. The proof will be divided into three steps as in the three lemmas below.

For a  $C^1$  solution  $(I, \rho, u)$ , we know that  $\rho - \bar{\rho}$  is supported in  $B(t) = \{x \in \mathbb{R}^3 : |x| \leq R + \sigma t\}$  by proposition 2.1. So, we can define

$$P(r, t) = \int_{|x|>r} \omega(x, r)(\rho(x, t) - \bar{\rho}) dx, \quad r > 0, \tag{3.1}$$

where

$$\omega(x, r) = |x|^{-1}(|x| - r)^2.$$

**Lemma 3.1.**

$$P(r, t) \geq P^0(r, t) + \frac{1}{2\sigma} \int_0^t \int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)} G(y, \tau) dy d\tau, \tag{3.2}$$

where

$$P^0(r, t) = \frac{1}{2} \left( q^0(r + \sigma t) + q^0(r - \sigma t) + \frac{1}{\sigma} \int_{r-\sigma t}^{r+\sigma t} q^1(y) dy \right), \tag{3.3}$$

$$G(r, t) = \int_{|x|>r} 2|x|^{-1} (p - \bar{p} - \sigma^2(\rho - \bar{\rho})) dx. \tag{3.4}$$

**Proof.** By direct computation, we have, using the first equation of (2.3),

$$\begin{aligned} \frac{\partial}{\partial t} P(r, t) &= \int_{|x|>r} \omega(x, r) \frac{\partial}{\partial t} (\rho(x, t) - \bar{\rho}) dx \\ &= - \int_{|x|>r} \omega(x, r) \nabla \cdot \rho u(x, t) dx \\ &= \int_{|x|>r} \nabla \omega(x, r) \cdot \rho u(x, t) dx, \end{aligned}$$

since  $\rho u$  is supported in  $B(t)$  and  $\omega(x, r) = 0$  when  $|x| = r$ . Thus,  $P(r, t)$  is  $C^2$  in  $t$ , and we can differentiate it again using (2.3):

$$\begin{aligned} \frac{\partial^2}{\partial t^2} P(r, t) &= \int_{|x|>r} \nabla \omega(x, r) \cdot \frac{\partial}{\partial t} (\rho u)(x, t) dx \\ &= - \sum_{i,j} \int_{|x|>r} \frac{\partial \omega}{\partial x_i} \frac{\partial}{\partial x_j} (\rho u_i u_j) dx - \int_{|x|>r} \nabla \omega \cdot \nabla (p - \bar{p}) dx \\ &\quad + \frac{1}{c} \int_{|x|>r} \int_0^\infty dv \int_{S^2} \nabla \omega \cdot \Omega(a(v, \rho)(I - \bar{B})) d\Omega dx, \end{aligned}$$

where  $\bar{p} = p(\bar{\rho})$ . Now since

$$\nabla \omega(x, r) = |x|^{-3}(|x|^2 - r^2)x,$$

which vanishes on  $\{x : |x| = r\}$ , and since  $\rho u_i u_j$  and  $p - \bar{p}$  have compact support, we integrate by parts again to obtain

$$\begin{aligned} \frac{\partial^2}{\partial t^2} P(r, t) &= \sum_{i,j} \int_{|x|>r} \frac{\partial^2 \omega}{\partial x_i \partial x_j} \rho u_i u_j dx + \int_{|x|>r} \Delta \omega \cdot (p - \bar{p}) dx \\ &\quad + \frac{1}{c} \int_{|x|>r} \int_0^\infty dv \int_{S^2} \nabla \omega \cdot \Omega(a(v, \rho)(I - \bar{B})) d\Omega dx \tag{3.5} \\ &\equiv I_1(r, t) + I_2(r, t) + I_3(r, t). \end{aligned}$$

A simple computation of  $\frac{\partial^2 \omega}{\partial x_i \partial x_j}$  shows that

$$I_1(r, t) = \int_{|x|>r} \frac{2r^2}{|x|^3} \cdot \rho \cdot \left( \frac{x}{|x|} \cdot u \right)^2 dx - \int_{|x|>r} \frac{|x|^2 - r^2}{|x|^3} \cdot \rho \cdot \left( \frac{x}{|x|} \cdot u \right)^2 dx + \int_{|x|>r} \frac{|x|^2 - r^2}{|x|^3} \cdot \rho \cdot |u|^2 dx \geq 0, \tag{3.6}$$

since  $\left(\frac{x}{|x|} \cdot u\right)^2 \leq |u|^2$ .

According to (2.10), we have  $I_3(r, t) \geq 0$ .

For the second term  $I_2$ , since

$$\Delta \omega(x, r) = 2|x|^{-1} = \omega_{rr}(x, r),$$

$$I_2(r, t) = \int_{|x|>r} 2|x|^{-1}(p - \bar{p}) dx = \frac{\partial^2}{\partial r^2} \int_{|x|>r} \omega(x, r)(p - \bar{p}) dx, \tag{3.7}$$

because  $\omega$  and  $\omega_r$  vanish on  $\{x : |x| = r\}$ . Combination of (3.5)–(3.7) gives

$$\left( \frac{\partial^2}{\partial t^2} - \sigma^2 \frac{\partial^2}{\partial r^2} \right) P(r, t) \geq G(r, t), \tag{3.8}$$

where

$$G(r, t) \equiv \frac{\partial^2}{\partial r^2} \int_{|x|>r} \omega(x, r) (p - \bar{p} - \sigma^2(\rho - \bar{\rho})) dx \equiv \frac{\partial^2}{\partial r^2} \tilde{G}(r, t). \tag{3.9}$$

We may also write

$$G(r, t) = \int_{|x|>r} 2|x|^{-1} (p - \bar{p} - \sigma^2(\rho - \bar{\rho})) dx.$$

Inversion of the one-dimensional d'Alembertian  $\square = \frac{\partial^2}{\partial t^2} - \sigma^2 \frac{\partial^2}{\partial r^2}$  in (3.8) for  $r > R_0 + \sigma t$  yields

$$P(r, t) = P^0(r, t) + \frac{1}{2\sigma} \int_0^t \int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)} \square P(y, \tau) dy d\tau \geq P^0(r, t) + \frac{1}{2\sigma} \int_0^t \int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)} G(y, \tau) dy d\tau.$$

The proof of lemma 3.1 is complete. □

Now let

$$F(t) = \int_0^t (t - \tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} r^{-1} P(r, \tau) dr d\tau. \tag{3.10}$$

**Lemma 3.2.**

$$F''(t) \geq C \frac{\sigma^4}{\bar{\rho}} \left( (\sigma t + R)^3 \ln \left( \frac{\sigma t + R}{R} \right) \right)^{-1} F(t)^2, \quad t \geq R_1, \tag{3.11}$$

$$F''(t) \geq B_0(\sigma t + R)^{-1}, \quad t > 0, \tag{3.12}$$

$$F'(t) \geq \sigma^{-1} B_0 \ln \left( \frac{\sigma t + R}{R} \right), \quad t > 0, \tag{3.13}$$

$$F(t) \geq C \sigma^{-2} B_0 (\sigma t + R) \ln \left( \frac{\sigma t + R}{R} \right), \quad t > R_1, \tag{3.14}$$

where

$$R_1 = \frac{1}{2\sigma} (R - R_0), \quad B_0 = \int_{R_0}^R q^0(r) dr.$$

**Proof.** Since  $F(t)$  is  $C^2$ , we have, from (3.2),

$$\begin{aligned} F''(t) &= \int_{\sigma t + R_0}^{\sigma t + R} r^{-1} P(r, t) \, dr \\ &\geq \int_{\sigma t + R_0}^{\sigma t + R} r^{-1} P^0(r, t) \, dr + \frac{1}{2\sigma} \int_{\sigma t + R_0}^{\sigma t + R} r^{-1} \int_0^t \int_{r - \sigma(t - \tau)}^{r + \sigma(t - \tau)} G(y, \tau) \, dy \, d\tau \, dr \\ &\equiv J_1 + J_2. \end{aligned} \quad (3.15)$$

By our hypotheses (2.8),  $q^0(r) > 0$  and  $q^1(r) \geq 0$  on  $R_0 < r < R$ . Hence, we see from (3.3) that

$$2J_1 \geq \int_{\sigma t + R_0}^{\sigma t + R} r^{-1} q^0(r - \sigma t) \, dr \geq (\sigma t + R)^{-1} \int_{\sigma t + R_0}^{\sigma t + R} q^0(r - \sigma t) \, dr = B_0(\sigma t + R)^{-1} > 0. \quad (3.16)$$

To bound  $J_2$  from below, we note that

$$p - \bar{p} - \sigma^2(\rho - \bar{\rho}) = Ae^{\bar{S}}(\rho^2 - \bar{\rho}^2 - 2\bar{\rho}(\rho - \bar{\rho})) = Ae^{\bar{S}}(\rho - \bar{\rho})^2 \geq 0. \quad (3.17)$$

It follows from (3.4) that

$$G(r, t) \geq 0.$$

In order to estimate  $J_2$  from below, we write it as

$$\begin{aligned} J_2 &= \frac{1}{2\sigma} \int_0^{t - R_1} \int_{\sigma\tau + R_0}^{\sigma\tau + R} G(y, \tau) \int_{\sigma t + R_0}^{y + \sigma(t - \tau)} r^{-1} \, dr \, dy \, d\tau \\ &\quad + \frac{1}{2\sigma} \int_{t - R_1}^t \int_{\sigma\tau + R_0}^{2\sigma t - \sigma\tau + R_0} G(y, \tau) \int_{\sigma t + R_0}^{y + \sigma(t - \tau)} r^{-1} \, dr \, dy \, d\tau \\ &\quad + \frac{1}{2\sigma} \int_{t - R_1}^t \int_{2\sigma t - \sigma\tau + R_0}^{\sigma\tau + R} G(y, \tau) \int_{y - \sigma(t - \tau)}^{y + \sigma(t - \tau)} r^{-1} \, dr \, dy \, d\tau \\ &\equiv J_2^1 + J_2^2 + J_2^3. \end{aligned}$$

In  $J_2^1$  we have

$$\begin{aligned} \int_{\sigma t + R_0}^{y + \sigma(t - \tau)} r^{-1} \, dr &\geq (y + \sigma(t - \tau))^{-1}(y - \sigma\tau - R_0) \\ &\geq (\sigma t + R)^{-1}(y - \sigma\tau - R_0) \\ &\geq C(\sigma t + R)^{-1} \left( \frac{t - \tau}{t} \right) (y - \sigma\tau - R_0)^2 \\ &\geq C\sigma(\sigma t + R)^{-2}(t - \tau)(y - \sigma\tau - R_0)^2. \end{aligned}$$

In  $J_2^2$  we have

$$\begin{aligned} \int_{\sigma t + R_0}^{y + \sigma(t - \tau)} r^{-1} \, dr &\geq C(\sigma t + R)^{-1}(y - \sigma\tau - R_0) \\ &\geq C\sigma(\sigma t + R)^{-2}(t - \tau)(y - \sigma\tau - R_0)^2 \end{aligned}$$

and in  $J_2^3$  we have

$$\begin{aligned} \int_{y - \sigma(t - \tau)}^{y + \sigma(t - \tau)} r^{-1} \, dr &\geq 2\sigma t(y + \sigma(t - \tau)) \\ &\geq C\sigma(\sigma t + R)^{-1}(t - \tau) \\ &\geq C\sigma(\sigma t + R)^{-2}(t - \tau)(y - \sigma\tau - R_0)^2. \end{aligned}$$

From the above estimates, it follows that

$$J_2 \geq C(\sigma t + R)^{-2} \int_0^t \int_{\sigma\tau+R_0}^{\sigma\tau+R} (t - \tau)(y - \sigma\tau - R_0)^2 G(y, \tau) dy d\tau,$$

for  $t \geq R_1$ . Returning to (3.9), and noting that  $\tilde{G}(y, \tau)$  vanishes for  $y > \sigma\tau + R$ , we can integrate by parts to obtain

$$\begin{aligned} J_2 &\geq C(\sigma t + R)^{-2} \int_0^t \int_{\sigma\tau+R_0}^{\sigma\tau+R} (t - \tau) \tilde{G}(y, \tau) dy d\tau \\ &\geq C(\sigma t + R)^{-2} \frac{\sigma^2}{\bar{\rho}} \int_0^t (t - \tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} \int_{|x|>y} \omega(\rho - \bar{\rho})^2 dx dy d\tau. \end{aligned} \quad (3.18)$$

Denoting this last integral by  $J_3$  and using Schwarz's inequality, we find

$$F^2(t) \leq J_3 \left( \int_0^t (t - \tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} \int_{y<|x|<\sigma\tau+R} \omega(x, y) dx dy d\tau \right). \quad (3.19)$$

Letting  $J_4$  denote the integral in (3.19), we have the following estimate:

$$\begin{aligned} J_4 &= \int_0^t (t - \tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} 4\pi \int_y^{\sigma\tau+R} |x|(|x| - y)^2 d|x| dy d\tau \\ &\leq C \int_0^t (t - \tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} (\sigma\tau + R)(\sigma\tau + R - y)^3 dy d\tau \\ &\leq C \int_0^t (t - \tau)(\sigma\tau + R) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} dy d\tau \leq C \int_0^t (t - \tau)(\sigma\tau + R)^{-1} d\tau \\ &\leq C\sigma^{-2}(\sigma\tau + R) \ln \left( \frac{\sigma t + R}{R} \right). \end{aligned}$$

Combining the above estimates and (3.15), we obtain

$$F''(t) \geq C \frac{\sigma^4}{\bar{\rho}} \left( (\sigma t + R)^3 \ln \left( \frac{\sigma t + R}{R} \right) \right)^{-1} F^2(t), \quad t \geq R_1,$$

since  $J_1 > 0$ . On the other hand, since  $J_2 \geq 0$ , (3.15) and (3.16) yield the estimates

$$F''(t) \geq B_0(\sigma t + R)^{-1}, \quad t > 0,$$

$$F'(t) \geq \sigma^{-1} B_0 \ln \left( \frac{\sigma t + R}{R} \right), \quad t > 0,$$

$$F(t) \geq C\sigma^{-2} B_0(\sigma t + R) \ln \left( \frac{\sigma t + R}{R} \right), \quad t > R_1,$$

since  $F(0) = F'(0) = 0$ . The proof of lemma 3.2 is complete. □

We now use lemma 3.2 to prove the finite-time blowup of  $C^1$  solutions.

**Lemma 3.3.** *The life span of the  $C^1$  solution is bounded above by*

$$\frac{C}{\sigma} \exp(C\sigma^2/B_0^2),$$

for some constant  $C > 0$ .

**Proof.** Indeed, if we let  $s = \sigma t$  and  $\tilde{F}(s) = \frac{\sigma^2}{\rho} F(t)$ , then we have  $\tilde{F}(0) = \tilde{F}'(0) = 0$ , and

$$\tilde{F}''(s) \geq C \left( (s + R)^3 \ln \left( \frac{s + R}{s} \right) \right)^{-1} \tilde{F}^2(s), \quad s > \frac{1}{2}(R - R_0) = k_1, \tag{3.20}$$

$$\tilde{F}'(s) \geq C \frac{B_0}{\rho} \ln \left( \frac{s + R}{s} \right), \quad s > 0, \tag{3.21}$$

$$\tilde{F}''(s) \geq \frac{B_0}{\rho} (s + R)^{-1}, \quad s > 0, \tag{3.22}$$

$$\tilde{F}(s) \geq C \frac{B_0}{\rho} (s + R) \ln \left( \frac{s + R}{s} \right), \quad s > k_1. \tag{3.23}$$

First, substitution of the lower bound in (3.23) into (3.20) yields

$$\tilde{F}''(s) \geq C \frac{B_0^2}{\rho^2} (s + R)^{-1} \ln \left( \frac{s + R}{s} \right), \quad s \geq k_1,$$

which leads to the improvement

$$\tilde{F}(s) \geq C \frac{B_0^2}{\rho^2} (s + R) \left[ \ln \left( \frac{s + R}{s} \right) \right]^2, \quad s \geq k_2 = 2R > k_1. \tag{3.24}$$

By (3.20) and interpolation with (3.24), we obtain

$$\tilde{F}''(s) \geq \mu(s) \tilde{F}(s), \quad s \geq k_2,$$

where

$$\mu(s) = C \frac{B_0^2}{\rho^2} (s + R)^{-2} \ln \left( \frac{s + R}{s} \right).$$

By (3.21),  $\tilde{F}'(s) \geq 0$ , so we can multiply the above inequality by  $\tilde{F}'(s)$  and integrate from  $k_3$  to  $s'$ , for any  $k_2 \leq k_3 < s'$ , to obtain

$$\tilde{F}'^2(s') \geq \tilde{F}'^2(k_3) + \int_{k_3}^{s'} \mu(s) (\tilde{F}^2(s))' ds.$$

Using integration by parts, we get

$$\tilde{F}'^2(s') \geq \tilde{F}'^2(k_3) + \mu(s') \tilde{F}^2(s') - \mu(k_3) \tilde{F}^2(k_3) - \int_{k_3}^{s'} \mu'(s) \tilde{F}^2(s) ds. \tag{3.25}$$

It is easy to check that  $\mu'(s) < 0$  for  $s \geq k_2$ , so the last term in (3.25) can be discarded. Since by (3.22),  $\tilde{F}''(s) > 0$ , and  $\tilde{F}(0) = 0$  we can deduce that

$$\tilde{F}(k_3) \leq k_3 \tilde{F}'(k_3).$$

Choose  $k_3$  so that  $k_3^2 \mu(k_3) \geq 1$ , i.e.

$$k_3 = O \left( \exp \left( \frac{C \rho^2}{B_0^2} \right) \right), \quad \text{as } B_0 \rightarrow 0. \tag{3.26}$$

Then, from (3.25),

$$\begin{aligned} \tilde{F}'^2(s') &\geq \tilde{F}'^2(k_3) + (k_3^2 \mu(k_3))^{-1} (\mu(s') \tilde{F}^2(s') - \mu(k_3) \tilde{F}^2(k_3)) \\ &\geq \frac{\mu(s')}{k_3^2 \mu(k_3)} \tilde{F}^2(s'). \end{aligned}$$

Since

$$\frac{\mu(s')}{k_3^2 \mu(k_3)} \geq C (s' + k)^{-2} \ln \left( \frac{s' + R}{R} \right),$$

we have

$$\tilde{F}'(s') \geq C(s' + R)^{-1} \left( \ln \left( \frac{s' + R}{R} \right) \right)^{1/2} \tilde{F}(s'), \quad s' \geq k_3.$$

Integrating this inequality, we obtain

$$\ln \left( \frac{\tilde{F}(s)}{\tilde{F}(k_3)} \right) \geq C \left( \ln \left( \frac{s + R}{k_3 + R} \right) \right)^{3/2}, \quad s \geq k_3.$$

So, if  $s \geq k_4 = Ck_3^2$ , we have,

$$\ln \left( \frac{\tilde{F}(s)}{\tilde{F}(k_3)} \right) \geq 8 \ln \left( \frac{s + R}{R} \right),$$

which by (3.24) leads to

$$\tilde{F}(s) \geq C \frac{B_0^2}{\rho^2} (s + R)^8, \quad s \geq k_4. \tag{3.27}$$

Thus, combining (3.20) and (3.27) we have

$$\tilde{F}''(s) \geq C \frac{B_0}{\rho} \tilde{F}^{3/2}(s), \quad s \geq k_4.$$

This inequality implies that, after integration as before,

$$\tilde{F}'^{1/2}(s) \geq C \frac{B_0}{\rho} (\tilde{F}^{5/2}(s) - \tilde{F}^{5/2}(k_4)).$$

Using the convexity of  $\tilde{F}(s)$ ,

$$\tilde{F}(s) \geq \tilde{F}'(k_4)(s - k_4) \geq \tilde{F}(k_4) \left( \frac{s - k_4}{k_4} \right).$$

So if  $s \geq k_5 = 3k_4$ , we obtain

$$\tilde{F}^{5/2}(s) - \tilde{F}^{5/2}(k_3) \geq \frac{1}{2} \tilde{F}^{5/2}(s).$$

Hence,

$$\tilde{F}'(s) \geq C \left( \frac{B_0}{\rho} \right)^{1/2} \tilde{F}^{5/4}(s), \quad s \geq k_5.$$

A final integration from  $k_5$  to  $T$ , shows

$$\frac{1}{\tilde{F}^{1/4}(k_5)} \geq C \left( \frac{B_0}{\rho} \right)^{1/2} T. \tag{3.28}$$

Therefore, the local existence time  $T$  is finite.

We have assumed, of course, that  $T > k_5$ . If  $T \leq k_5 = Ck_3^2$ , we see from (3.26) that  $T \leq \exp\left(\frac{C\bar{\rho}^2}{B_0^2}\right)$ , as  $B_0 \rightarrow 0$ . On the other hand, for  $T > k_5$ , (3.28) holds, and using (3.27) we see that

$$C \left( \frac{B_0}{\rho} \right)^{1/2} k_5 \leq \left( \frac{C}{k_5^8 (B_0/\bar{\rho})^2} \right)^{1/4},$$

i.e.  $C \frac{B_0}{\rho} \exp(C\bar{\rho}^2/B_0^2) \leq 1$ . As  $B_0 \rightarrow 0$ , this is impossible. Thus

$$T \leq \exp(C\bar{\rho}^2/B_0^2) = k_5$$

when  $B_0$  is small. Hence, the life span of the  $C^1$  solution is bounded above by

$$\frac{C}{\sigma} \exp(C\sigma^2/B_0^2).$$

The proof of lemma 3.3 is complete.  $\square$

For the general case,  $\gamma > 1$ , the adjustment is needed in (3.17) which now becomes

$$p - \bar{p} - \sigma^2(\rho - \bar{\rho}) \geq Ae^{\bar{s}} (\rho^\gamma - \bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho})) \equiv Ae^{\bar{s}}\Psi(\rho, \bar{\rho}). \quad (3.29)$$

Since  $\rho^\gamma$  is convex, then

$$\Psi(\rho, \bar{\rho}) = \rho^\gamma - \bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho}) > 0,$$

for  $\rho \neq \bar{\rho}$ . However, by Taylor's theorem one has

$$\Psi(\rho, \bar{\rho}) \geq C_0(\gamma, \bar{\rho})(\rho - \bar{\rho})^2,$$

for  $0 < \rho < 2\bar{\rho}$ , and some positive constant  $C_0(\gamma, \bar{\rho})$ . On the other hand, there exists a positive constant  $C_1(\gamma)$  such that

$$\Psi(\rho, \bar{\rho}) \geq C_1(\gamma)(\rho - \bar{\rho})^\gamma \quad \text{for } \rho \geq 2\bar{\rho}.$$

Therefore, there exists a positive constant  $C(\gamma, \bar{\rho})$  such that

$$\Psi(\rho, \bar{\rho}) \geq C(\gamma, \bar{\rho})\Phi_\gamma(\rho - \bar{\rho}),$$

where  $\Phi_\gamma$  is a nonnegative convex function given by

$$\Phi_\gamma(\rho - \bar{\rho}) = \begin{cases} (\rho - \bar{\rho})^2, & 0 < \rho < 2\bar{\rho}, \\ (\rho - \bar{\rho})^\gamma, & \rho \geq 2\bar{\rho}. \end{cases}$$

Finally, Jensen's inequality should be used instead of Schwarz's inequality in (3.19). The rest of the details should then follow accordingly, and the resulting differential inequality still has a finite life span. However, the upper bound for the local existence time  $T$  will be different from the one which we have obtained for the case  $\gamma = 2$ .

The proof of theorem 2.2 is complete.

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