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EVOLUTION OF DISCONTINUITY AND FORMATION OF TRIPLE-SHOCK PATTERN IN SOLUTIONS TO A TWO-DIMENSIONAL HYPERBOLIC SYSTEM OF CONSERVATION LAWS*

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Abstract. The evolution of discontinuity and formation of triple-shock pattern in solutions to a two-dimensional hyperbolic system of conservation laws are studied. When the initial discontinuity is a convex curve, it is discovered that the structure of the global solution changes dramatically around a critical time: After the critical time, a triple-shock pattern forms, while, before the critical time, only two shocks are developed. The envelope surface of intersections and the evolution of discontinuity are analyzed by developing new ideas and approaches. The global structure of the entropy solution is presented.

Key words. two-dimensional conservation laws, global structure of solutions, evolution of discontinuity, characteristic planes, envelope, formation of triple-shock pattern

AMS subject classifications. 35J65, 76G25, 35J70, 76N10

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1. Introduction.

We are interested in the global structure and the evolution of discontinuity of solutions to multidimensional hyperbolic systems of conservation laws. It is well known that the formation of singularities in solutions causes a major difficulty in solving hyperbolic systems of conservation laws (cf. [1, 9, 10]). The one-dimensional hyperbolic systems of conservation laws have been understood relatively well (cf. [6, 2, 5, 7] and the references therein), while the analysis of multidimensional systems is challenging and requires new techniques. One of the essential difficulties for general multidimensional problems is that we do not have enough knowledge on the structure of solutions to identify the function spaces for the solutions. In this paper, we study the Cauchy problem of the following two-dimensional system:

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0, \\ v_t + (uv)_x + (v^2)_y = 0, \end{cases} \tag{1.1}$$

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with initial data

$$(u,v)|_{t=0} = \begin{cases} (u_-, v_-) & \text{if } y-\phi(x)<0, \\ (u_+, v_+) & \text{if } y-\phi(x)>0, \end{cases} \tag{1.2}$$

where the initial discontinuity $y-\phi(x)=0$ is a smooth curve that divides the $x-y$ plane into two parts with constant states $(u_\pm, v_\pm)$. System (1.1) arises in magnetohydrodynamics, elasticity theory, and oil recovery (cf. [14]). There have been many studies on system (1.1) from various aspects; see [3, 8, 11, 12, 13, 15, 16] and the references cited therein. Our study in this paper is to construct explicitly a global solution $U=(u,v)$ of (1.1)–(1.2) and to investigate the evolution of discontinuity and the formation of triple-shock pattern. If initial discontinuity $y=\phi(x)$ is a straight line, this Cauchy problem can be reduced to a one-dimensional Riemann problem, and the solution is self-similar. When the initial discontinuity $y=\phi(x)$ is a curve, we have a truly two-dimensional problem that is not self-similar: The solution is not self-similar, the two-dimensional elementary waves are not self-similar, and the intermediate states cannot be constant. We are interested in the case where $y=\phi(x)$ is a curve, and thus new ideas will be developed to construct intermediate state $U_m=(u_m,v_m)$, as well as the discontinuity surfaces connecting left state $U_-=(u_-,v_-)$, intermediate state $U_m=(u_m,v_m)$, and right state $U_+=(u_+,v_+)$. For the self-similar solutions of two-dimensional Riemann problems, see [17] and the references therein.

We first identify the characteristic planes such that the intermediate state is constant on each characteristic plane. The constants are different on different characteristic planes. Then we need to find one discontinuity surface that connects the left state with the intermediate state, and the other discontinuity surface that connects the intermediate state with the right state. For Cauchy problem (1.1)–(1.2), we see that the discontinuity surface connecting the left state and the intermediate state is a contact discontinuity. As for the discontinuity surface connecting the intermediate state and the right state, if the initial discontinuity curve $y=\phi(x)$ is concave down, i.e., $\phi''(x)<0$, the characteristic planes do not intersect, and this discontinuity surface is a single shock, as shown in [15, 16]. When the initial discontinuity curve $y=\phi(x)$ is convex, i.e., $\phi''(x)>0$, the characteristic planes intersect, which makes the problem more complicated. In this case, we need to study the envelope of the intersection points of the characteristic planes, especially the shape and the cusp of the envelope. Then we explore the possible discontinuity surfaces connecting the intermediate state and the right state. We find that there exists a critical time such that, before the critical time, only two shocks are developed; however, after the critical time, the triple-shock pattern forms. Therefore, the structure of the solution changes dramatically around the critical time. A numerical result by Chou and Shu [4] also shows the same phenomenon as illustrated in Figures 6–7 below. The analysis will be carried out to prove the evolution of the discontinuity, and the global structure of the entropy solution will be provided explicitly. We remark that many problems are still open in this direction such as the global structure of solutions in the case of multicusps of envelope, rarefaction waves, closed curve of initial discontinuity, as well as the generalization to the original Euler equations. Our preliminary analysis shows that these cases are much more complicated.

The rest of the paper is organized as follows. In section 2, we state our main results. In section 3, we provide some basic properties of system (1.1), including the jump conditions, entropy conditions, characteristic planes, intermediate states, and elementary waves. In section 4, we study the envelope surface, the cusp, and its shape. In section 5, we study the inner shock surface developed in the region bounded by the

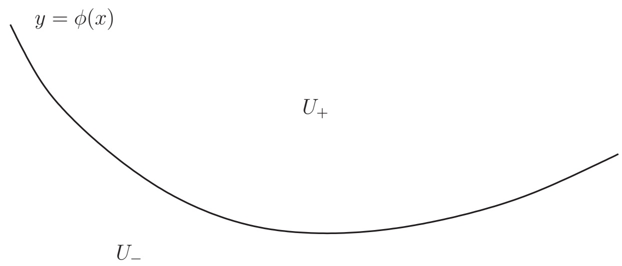

FIG. 1. *Initial discontinuity.*

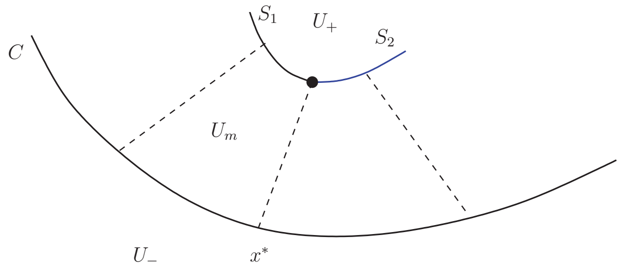

FIG. 2. *Structure of solution before the critical time.*

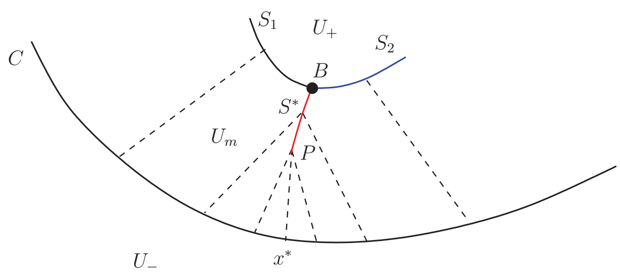

FIG. 3. *Triple-shock pattern after the critical time.*

envelope. In section 6, we show that, before the critical time, the discontinuity surface connecting the intermediate state and the right state consists of two shocks. In section 7, we study the interaction of the inner shock discussed in section 5 and the shocks analyzed in section 6. Finally, in section 8, we study the formation of triple-shock pattern.

2. Main results. In this section, we state our main results. Consider the case where initial discontinuity curve $y = \phi(x)$ is convex, i.e., $\phi''(x) > 0$ for all $x \in \mathbb{R}$. In this case, we will see later that the characteristic planes intersect. We will study envelope surface Π of the intersections, as well as cusp curve $\widehat{PP_*}$. We will prove that the discontinuity surface connecting left state $U_- = (u_-, v_-)$ with intermediate state $U_m = (u_m, v_m)$ is a contact discontinuity, while the discontinuity connecting intermediate state (u_m, v_m) with right state $U_+ = (u_+, v_+)$ is completely different before and after a critical time $T > 0$. Before critical time T, there are two shocks S_1 and S_2; after time T, a triple-shock pattern forms with two shocks S_1, S_2, and an additional inner shock S^* inside the region bounded by two branches Π_1, Π_2 of the envelope.

Figures 1–3 show the evolution of initial discontinuity $y = \phi(x)$: At $t = 0$, the discontinuity is a convex curve $y = \phi(x)$, as in Figure 1; for $0 < t < T$, there are two

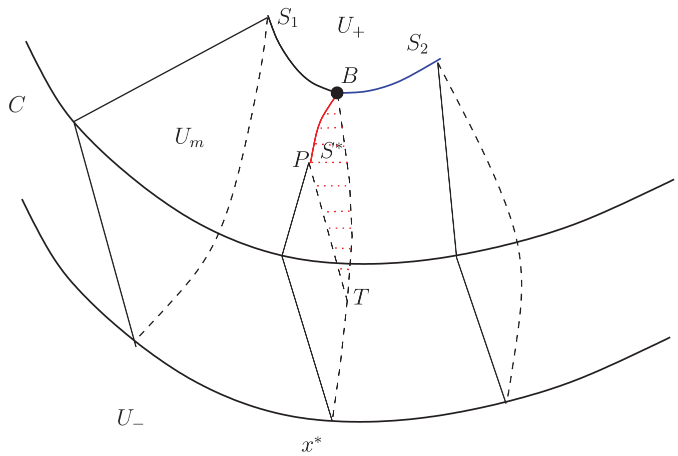

FIG. 4. *A global solution with triple-shock pattern in space and time.*

shocks S_1 and S_2 and a contact discontinuity C, as in Figure 2; for $t > T$, besides the contact discontinuity C and two shocks S_1 and S_2, the third shock S^* develops, and the triple-shock pattern forms at triple point B, as in Figure 3 (also see Figure 4).

Denote

$$n_-(x) := -\phi'(x)u_- + v_-, \qquad n_+(x) := -\phi'(x)u_+ + v_+.$$

Then the main result of this paper is as follows.

THEOREM 2.1. *For Cauchy problem* (1.1)–(1.2), *assume that* $J = u_+v_- - u_-v_+ \neq 0$ *and function* $\phi \in C^3(\mathbb{R})$ *satisfies the following conditions:*

(i) $\phi''(x) > 0$ *and* $n_-(x) > n_+(x) > 0$ *on* $\mathbb{R}$*;*
(ii) *there exist some* $x^0 \in \mathbb{R}$ *such that* $n_-(x^0) < 2n_+(x^0)$*;*
(iii) $G(x) := n_-(x)\phi'''(x) + 3u_-(\phi''(x))^3$ *has a unique zero point* $x^* \in \mathbb{R}$ *and* $G'(x^*) < 0$*;*
(iv) $H(x) := u_+u_-\phi'(x) - (2u_+v_- - u_-v_+)$ *has a unique zero point* $x^{**} \in \mathbb{R}$.

Then

(a) *there exist characteristic planes such that intermediate state* $U_m = (u_m, v_m)$ *is constant on each characteristic plane;*
(b) *the discontinuity surface connecting left state* $U_- = (u_-, v_-)$ *with intermediate state* $U_m = (u_m, v_m)$ *is a contact discontinuity;*
(c) *there exists a critical time* $T > 0$ *such that the discontinuity surface connecting intermediate state* $U_m = (u_m, v_m)$ *with right state* $U_+ = (u_+, v_+)$ *is completely different before and after time* T*. Before critical time* T*, there are only two shocks* S_1 *and* S_2*, while after critical time* T*, a triple-shock pattern forms which consists of three shocks* S_1*,* S_2*, and additional inner shock* S^* *inside the region bounded by two branches* Π_1*,* Π_2 *of the envelope surface* Π *of intersections of characteristic planes.*

We remark that the following example

$$\phi(x) = e^{-x}, \quad u_- > u_+ > 0, \quad v_- > v_+ > 0, \quad u_-v_+ > 2u_+v_-, \tag{2.1}$$

satisfies the conditions of Theorem 2.1. The global structure and evolution of discontinuity in space and time is sketched in Figure 4, where point B is a triple point

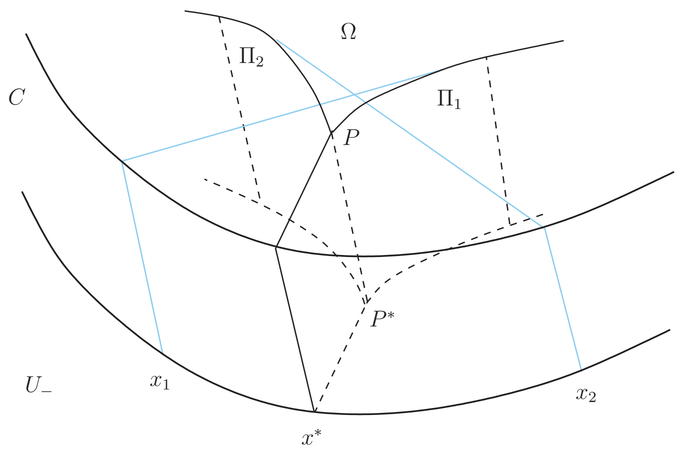

FIG. 5. *Envelope* $\Pi := \Pi_1 \cup \Pi_2$ *in space and time.*

with triple shocks S_1, S_2, S^* and triangle surface BPT is the additional inner shock surface developed after critical time T. Figure 5 shows the envelope surface Π and its two branches Π_1 and Π_2.

The numerical computations by Chou and Shu [4] also show the results in Theorem 2.1. Take

$$\phi(x) = e^{-x}, \qquad (u_-, v_-) = (200, 2), \qquad (u_+, v_+) = (10, 1),$$

which satisfies (2.1). The numeral pictures in Figures 6–7 show the contour curves of v (similar for u) at $t = 1$ and $t = 5$, respectively. Figure 6 shows the contact discontinuity on the left and two shocks on the right splitting into two sections as in Figure 2, and it seems that $t = 1$ is close to the critical time. Figure 7 shows the contact discontinuity on the left and the triple-shock pattern on the right. In Figures 6–7, it seems that the contact discontinuity terminates at a finite point, but this happens only because the value of v is too small after that point to appear in the figures due to the choice of $\phi(x)$. We also note that the scales in Figures 6–7 are different, and the triple shock in Figure 7 occurs where both x and y are quite big.

The rest of the paper is devoted to the proof of Theorem 2.1.

3. Basic properties of system (1.1). We rewrite system (1.1) as

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 2u & 0 \\ v & u \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x + \begin{bmatrix} v & u \\ 0 & 2v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_y = 0 \tag{3.1}$$

and set

$$A := \begin{bmatrix} 2u & 0 \\ v & u \end{bmatrix}, \quad B := \begin{bmatrix} v & u \\ 0 & 2v \end{bmatrix}.$$

In a direction (α, β) (with $\alpha^2 + \beta^2 = 1$), the roots of $\det(\lambda I - \alpha A - \beta B) = 0$ are the eigenvalues of system (3.1), equivalently (1.1),

$$\lambda_1^{(\alpha,\beta)} = \alpha u + \beta v = (u, v) \cdot (\alpha, \beta), \qquad \lambda_2^{(\alpha,\beta)} = 2(\alpha u + \beta v) = 2(u, v) \cdot (\alpha, \beta),$$

and the corresponding right eigenvectors are

$$r_1 = (\beta, -\alpha), \qquad r_2 = (u, v).$$

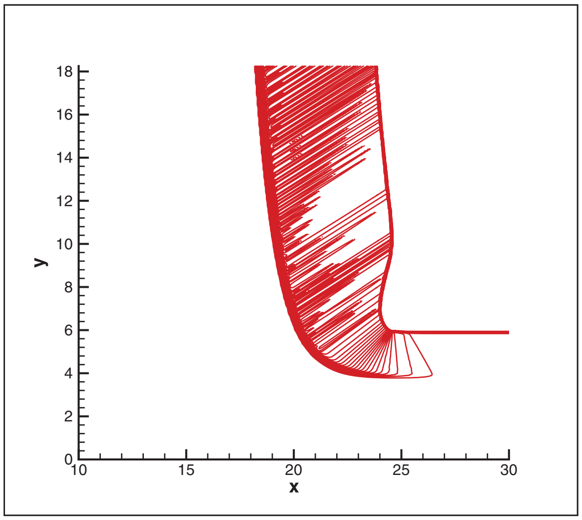

FIG. 6. *The solution structure before the critical time.*

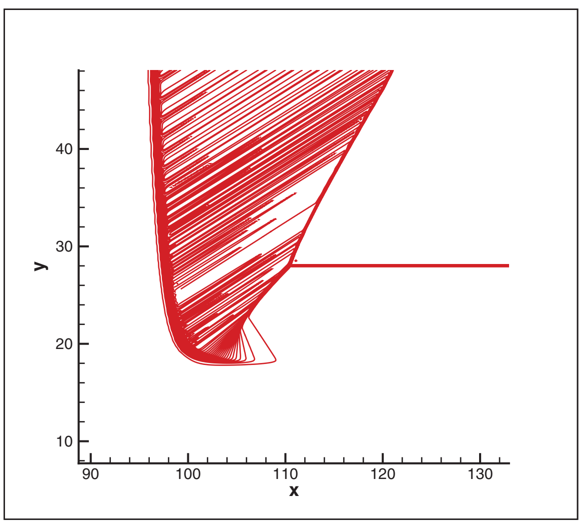

FIG. 7. *Triple-shock pattern after the critical time.*

We notice the linear degeneracy of the first characteristic field:

$$\nabla\lambda_1 \cdot r_1 = 0,$$

while the second characteristic field satisfies

$$\nabla\lambda_2 \cdot r_2 = 2(\alpha u + \beta v).$$

3.1. Two-dimensional Rankine–Hugoniot conditions. Let us assume that $S(x, y, t) = 0$ is a surface of discontinuity of a solution to system (1.1) and (u_l, v_l) and (u_r, v_r) are the values on the side $S(x, y, t) < 0$ and the side $S(x, y, t) > 0$, respectively. Then the Rankine–Hugoniot conditions are

$$\begin{cases} [u]S_t + [u^2]S_x + [uv]S_y = 0, \\ [v]S_t + [uv]S_x + [v^2]S_y = 0, \end{cases} \tag{3.2}$$

where $[w] = w_r - w_l$ denotes the jump of function w across discontinuity surface $S = 0$.

LEMMA 3.1. *Rankine–Hugoniot condition* (3.2) *holds if and only if*

$$\begin{cases} S_t + u_l S_x + v_l S_y = 0, \\ [u]S_x + [v]S_y = 0, \end{cases} \tag{3.3}$$

or

$$\begin{cases} S_t + (u_l + u_r)S_x + (v_l + v_r)S_y = 0, \\ \frac{u_l}{v_l} = \frac{u_r}{v_r}. \end{cases} \tag{3.4}$$

This can be seen as follows: If (3.2) holds, then

$$\begin{bmatrix} S_t + (u_l + u_r)S_x + v_l S_y & u_r S_y \\ v_r S_x & S_t + u_l S_x + (v_l + v_r)S_y \end{bmatrix} \begin{bmatrix} [u] \\ [v] \end{bmatrix} = 0. \tag{3.5}$$

Since $([u], [v]) \neq (0, 0)$, the matrix in (3.5) is singular, that is,

$$(S_t + (u_l + u_r)S_x + v_l S_y)\,(S_t + u_l S_x + (v_l + v_r)S_y) - u_r v_r S_x S_y = 0,$$

which yields

$$S_t + u_l S_x + v_l S_y = 0$$

or

$$S_t + (u_l + u_r)S_x + (v_l + v_r)S_y = 0.$$

The second equation in (3.3) or (3.4) follows from the first equation and (3.2). It is easy to check that (3.2) holds if (3.3) or (3.4) holds.

3.2. Contact discontinuities, shocks, and entropy conditions. Denote the normal vector on the $x - y$ plane of the discontinuity curve $S(x, y, t) = 0$, with t fixed by

$$n := \frac{(S_x, S_y)}{\sqrt{S_x^2 + S_y^2}}$$

and the eigenvalues along the normal direction by

$$\lambda_{1n}^l = (u_l, v_l) \cdot n = \frac{u_l S_x + v_l S_y}{\sqrt{S_x^2 + S_y^2}}, \qquad \lambda_{1n}^r = (u_r, v_r) \cdot n = \frac{u_r S_x + v_r S_y}{\sqrt{S_x^2 + S_y^2}},$$

$$\lambda_{2n}^l = \frac{2(u_l S_x + v_l S_y)}{\sqrt{S_x^2 + S_y^2}}, \qquad \lambda_{2n}^r = \frac{2(u_r S_x + v_r S_y)}{\sqrt{S_x^2 + S_y^2}}.$$

At any point $(x(t), y(t), t)$ on discontinuity surface $S(x, y, t) = 0$, the discontinuity propagates with velocity $(x'(t), y'(t))$ on the $x - y$ plane. The propagation speed in the normal direction is denoted by

$$V_n^s := (x'(t), y'(t)) \cdot n = -\frac{S_t}{\sqrt{S_x^2 + S_y^2}},$$

using $S_x x'(t) + S_y y'(t) + S_t = 0$.

DEFINITION 3.1. *Discontinuity surface $S = 0$ is called a k-*shock *($k = 1$ or 2), denoted by S, if* (3.4) *and the following entropy conditions hold:*

$$\lambda_{kn}^r < V_n^s < \lambda_{kn}^l, \quad \lambda_{(k-1)n}^l < V_n^s < \lambda_{(k+1)n}^r. \tag{3.6}$$

*Discontinuity surface $S = 0$ is called a k-*contact discontinuity*, denoted by C, if* (3.3) *holds and $\lambda_{kn}^r = V_n^s = \lambda_{kn}^l$.*

3.3. Connection between (u_-, v_-) and intermediate state (u_m, v_m). Denote the intermediate state between left state (u_-, v_-) and right state (u_+, v_+) by (u_m, v_m).

LEMMA 3.2. *The two-dimensional discontinuity surface connecting (u_-, v_-) and intermediate state (u_m, v_m) must be a 1-contact discontinuity.*

Proof. Suppose that the discontinuity surface connecting (u_-, v_-) with intermediate state (u_m, v_m) is $S_1(x, y, t) = 0$, which would be a two-dimensional 1-shock. According to Lemma 3.1 and Definition 3.1, $S_1(x, y, t)$ must satisfy either

$$\begin{cases} S_{1t} + u_- S_{1x} + v_- S_{1y} = 0, \\ u_- S_{1x} + v_- S_{1y} = u_m S_{1x} + v_m S_{1y}, \end{cases} \tag{3.7}$$

or

$$\begin{cases} S_{1t} + (u_- + u_m) S_{1x} + (v_- + v_m) S_{1y} = 0, \\ \begin{vmatrix} u_- & v_- \\ u_m & v_m \end{vmatrix} = 0, \end{cases} \tag{3.8}$$

and (3.6). From (3.6), we have

$$u_- S_{1x} + v_- S_{1y} + S_{1t} > 0 > u_m S_{1x} + v_m S_{1y} + S_{1t} \tag{3.9}$$

and

$$2u_m S_{1x} + 2v_m S_{1y} + S_{1t} > 0. \tag{3.10}$$

By (3.9), one has

$$u_- S_{1x} + v_- S_{1y} > u_m S_{1x} + v_m S_{1y}.$$

Thus, $S_1(x, y, t)$ satisfies (3.8) instead of (3.7), i.e.,

$$S_{1t} + (u_- + u_m) S_{1x} + (v_- + v_m) S_{1y} = 0. \tag{3.11}$$

From (3.9) and (3.11), we have

$$u_- S_{1x} + v_- S_{1y} + S_{1t} > S_{1t} + (u_- + u_m) S_{1x} + (v_- + v_m) S_{1y}$$

and

$$S_{1t} + (u_- + u_m)S_{1x} + (v_- + v_m)S_{1y} > u_m S_{1x} + v_m S_{1y} + S_{1t},$$

which yield

$$u_- S_{1x} + v_- S_{1y} > 0 > u_m S_{1x} + v_m S_{1y}.$$

Hence, we have

$$\begin{aligned} 2u_m S_{1x} + 2v_m S_{1y} + S_{1t} &= u_m S_{1x} + v_m S_{1y} + S_{1t} + (u_m S_{1x} + v_m S_{1y}) \\ &< u_m S_{1x} + v_m S_{1y} + S_{1t} < 0, \end{aligned}$$

which contradicts with (3.10). Therefore, S_1 must be a 1-contact discontinuity. ∎

LEMMA 3.3 (contact discontinuity). *Contact discontinuity surface $S(x,y,t) = 0$ connecting (u_-, v_-) with intermediate state (u_m, v_m) is given by the equation*

$$y - v_- t - \phi(x - u_- t) = 0. \tag{3.12}$$

Proof. According to Lemma 3.1, contact discontinuity surface $S(x,y,t) = 0$ satisfies

$$S_t + u_- S_x + v_- S_y = 0$$

and

$$S(x,y,0) = y - \phi(x).$$

This implies

$$S(x,y,t) = y - v_- t - \phi(x - u_- t) = 0, \tag{3.13}$$

which is the equation of the surface of contact discontinuity. ∎

Contact discontinuity surface (3.12) is a cylindrical surface, and

$$\begin{cases} x = x_0 + u_- t, \\ y = \phi(x_0) + v_- t, \\ t = t, \end{cases} \tag{3.14}$$

is its generator corresponding to $x_0 \in \mathbb{R}$, with parameter $t \geq 0$. We call (3.14) an x_0-*generator*.

Remark 3.1. Contact discontinuity surface (3.13) should also satisfy the second equation in (3.3), i.e.,

$$u_r S_x + v_r S_y = u_- S_x + v_- S_y, \tag{3.15}$$

where (u_r, v_r) is the value of (u, v) on the intermediate side of contact discontinuity $S(x,y,t) = 0$ and will be determined in the next subsection.

3.4. Construction of (u_r, v_r) and intermediate state (u_m, v_m). From Definition 3.1 and Lemma 3.1, we see that intermediate state (u_m, v_m) must satisfy that $\frac{u_m}{v_m} = \frac{u_+}{v_+}$ on the interface connecting (u_m, v_m) and (u_+, v_+). Thus, it is natural to construct intermediate state (u_m, v_m) satisfying

$$\frac{u_m}{v_m} = \frac{u_+}{v_+}, \tag{3.16}$$

which also implies that

$$\frac{u_r}{v_r} = \frac{u_+}{v_+}. \tag{3.17}$$

Here we recall that (u_r, v_r) is the value of (u_m, v_m) on contact discontinuity $S(x, y, t) = y - v_-t - \phi(x - u_-t) = 0$ and should satisfy (3.15). We note that, from (3.17),

$$u_r S_x + v_r S_y = u_r \left(S_x + \frac{v_+}{u_+} S_y \right) = \frac{u_r}{u_+}(u_+ S_x + v_+ S_y),$$

which, together with (3.15), yields

$$u_r = u_+ \frac{u_- S_x + v_- S_y}{u_+ S_x + v_+ S_y}, \quad v_r = \frac{v_+}{u_+} u_r = v_+ \frac{u_- S_x + v_- S_y}{u_+ S_x + v_+ S_y}.$$

Since any point (x, y, t) on contact discontinuity $S(x, y, t) = 0$ must be also on the certain x_0-generator in (3.14) for some $x_0 \in \mathbb{R}$, we see that

$$S_x = -\phi'(x - u_-t) = -\phi'(x_0), \quad S_y = 1. \tag{3.18}$$

Thus, on x_0-generator (3.14), (u_r, v_r) is constant given by

$$u_r = u_+ N(x_0), \quad v_r = v_+ N(x_0), \tag{3.19}$$

where

$$N(x) := \frac{n_-(x)}{n_+(x)}, \quad n_\pm(x) := -\phi'(x) u_\pm + v_\pm.$$

We now construct intermediate state (u_m, v_m). For (u_m, v_m), by (3.16), system (1.1) reduces to the scalar equation

$$(u_m)_t + (u_m^2)_x + \left(\frac{v_+}{u_+} u_m^2 \right)_y = 0, \tag{3.20}$$

with characteristic direction

$$\left(2u_m, 2\frac{v_+}{u_+} u_m, 1 \right) = (2u_m, 2v_m, 1).$$

Along this characteristic direction, u_m and thus v_m are constant. On the characteristics starting from x_0-generator (3.14),

$$u_m = u_r = u_+ N(x_0), \quad v_m = v_r = v_+ N(x_0). \tag{3.21}$$

Thus, all the characteristics start from the x_0-generator form a semicharacteristic plane corresponding to x_0, called an x_0-plane, which is determined by point

$(x_0, \phi(x_0), 0)$ and two vectors $(u_-, v_-, 1)$ and $(2u_r, 2v_r, 1) = (2u_+N(x_0), 2v_+N(x_0), 1)$ starting at $(x_0, \phi(x_0), 0)$. On the x_0-plane, both u_m and v_m are constants which depend on x_0 as in (3.21). When $(x_0, \phi(x_0), 0)$ continuously moves along initial discontinuity curve $y - \phi(x) = 0$, we obtain a family of such x_0-planes. Intermediate state

$$(u_m, v_m) = (u_+N(x_0), v_+N(x_0)) = (u_+N(x - u_-t), v_+N(x - u_-t))$$

given in (3.21) is a smooth solution of system (1.1).

For any point (x, y, t) on the x_0-plane, we note that three vectors

$$(x - x_0, y - \phi(x_0), t)\,, \quad (2u_+N(x_0), 2v_+N(x_0), 1)\,, \quad (u_-, v_-, 1)\,,$$

are all on the same x_0-plane, thus

$$F(x, y, t, x_0) := \det \begin{bmatrix} x - x_0 & y - \phi(x_0) & t \\ 2u_+N(x_0) & 2v_+N(x_0) & 1 \\ u_- & v_- & 1 \end{bmatrix} = 0. \tag{3.22}$$

The simple calculations show that

$$\begin{aligned} F(x, y, t, x_0) = &\,(x - u_-t - x_0)\,(2v_+N(x_0) - v_-) \\ &- (y - v_-t - \phi(x_0))\,(2u_+N(x_0) - u_-) \end{aligned} \tag{3.23}$$

and

$$F_{x_0}(x, y, t, x_0) = 2\frac{J\phi''(x_0)}{n_+^2(x_0)}\,(v_+(x - u_-t - x_0) - u_+(y - v_-t - \phi(x_0))) - n_-(x_0), \tag{3.24}$$

where

$$J := \det \begin{bmatrix} u_+ & v_+ \\ u_- & v_- \end{bmatrix} \neq 0.$$

Equation (3.22) is the equation of the characteristic plane corresponding to $x_0 \in \mathbb{R}$.

3.5. Connection between (u_m, v_m) and (u_+, v_+). We have the following lemma about this connection between (u_m, v_m) and (u_+, v_+).

LEMMA 3.4. *If $n_- > n_+$ on $\mathbb{R}$, then the elementary wave connecting (u_m, v_m) and (u_+, v_+) is a 2-shock.*

Proof. The elementary wave connecting (u_m, v_m) and (u_+, v_+) is a 2-wave. An argument similar to that in Lemma 3.2 shows that it must be a 2-shock. We omit the details of the proof. ☐

3.6. The case of $\phi'' < 0$. In the case of $\phi'' < 0$, the characteristic planes do not intersect; $x_0 = x_0(x, y, t)$ can be defined globally as an implicit function in the region $y - v_-t - \phi(x - u_-t) > 0$ through (3.22). The connection between (u_m, v_m) and (u_+, v_+) is a single shock, while the connection between (u_-, v_-) and (u_m, v_m) is contact discontinuity (3.13). We record the following proposition from [16].

PROPOSITION 3.1 (see [16]). *If $\phi'' < 0$ and $n_- > n_+ > 0$ on $\mathbb{R}$, then the global solution of* (1.1)–(1.2) *is the following:*

$$(u, v)(x, y, t) = \begin{cases} (u_-, v_-) & \text{if } y - v_-t - \phi(x - u_-t) < 0, \\ (u_+N(x_0), v_+N(x_0)) & \text{if } y - v_-t - \phi(x - u_-t) > 0 \text{ and } \bar{S}(x, y, t) < 0, \\ (u_+, v_+) & \text{if } \bar{S}(x, y, t) > 0, \end{cases}$$

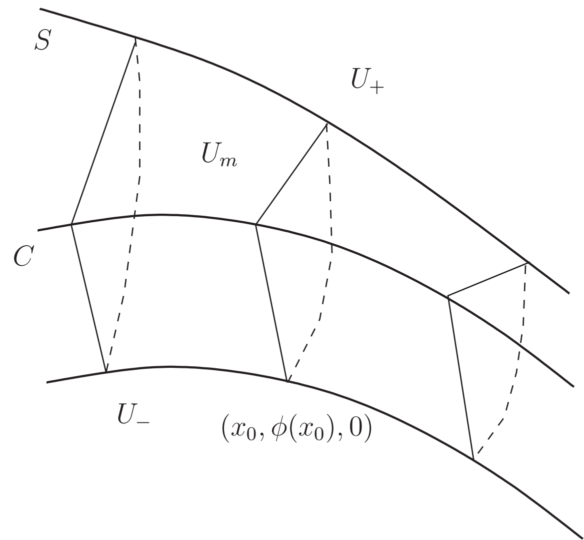

FIG. 8. *Global structure of solution when $\phi''(x) < 0$ in space and time.*

where $x_0 = x_0(x, y, t)$ is the global implicit function satisfying $F(x, y, t, x_0) = 0$ and $\bar{S}(x, y, t)$ is the function satisfying

$$\begin{cases} \bar{S}_t + (u_+ N(x_0) + u_+)\bar{S}_x + (v_+ N(x_0) + v_+)\bar{S}_y = 0, \\ \bar{S}(x, y, 0) = y - \phi(x). \end{cases}$$

Remark 3.2. In Proposition 3.1, $y - v_- t - \phi(x - u_- t) = 0$ is the two-dimensional 1-contact discontinuity, $(u_+ N(x_0), v_+ N(x_0)) = (u_m, v_m)$ is the intermediate state, and $\bar{S}(x, y, t) = 0$ is a two-dimensional shock with the following parametric form:

$$x = x(\beta, t), \quad y = y(\beta, t), \quad t = t,$$

where $\beta \in (-\infty, \infty)$, $t \geq 0$, $(x(\beta, t), y(\beta, t))$ is the unique solution of following ordinary differential equations:

$$\begin{cases} \frac{dx}{dt} = u_+ N(x_0) + u_+, \\ \frac{dy}{dt} = v_+ N(x_0) + v_+, \\ x|_{t=0} = \beta, \\ y|_{t=0} = \phi(\beta). \end{cases}$$

Figure 8 sketches the global structure of the solution in Proposition 3.1. The triangle planes are the characteristic planes with no intersections in this case.

4. Envelope surface. When $\phi'' > 0$, characteristic planes $F(x, y, t, x_0) = 0$ intersect, as illustrated in Figure 9. Thus, we need to study the envelope of the intersection points of the characteristic planes.

4.1. Equations for the envelope surface. We recall that the equation of the characteristic plane associated with point $(x_0, \phi(x_0), 0)$, or the x_0-plane, is

$$F(x, y, t, x_0) = 0. \tag{4.1}$$

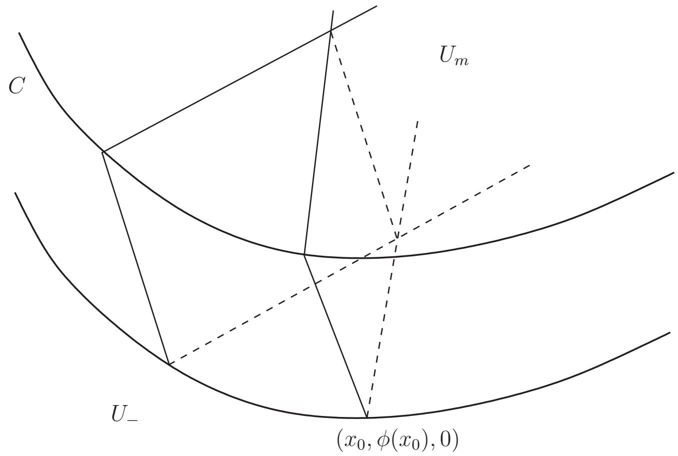

FIG. 9. *Characteristic planes intersect.*

The envelope surface Π of intersection is determined by the following equations:

$$\begin{cases} F(x,y,t,x_0) = 0, \\ F_{x_0}(x,y,t,x_0) = 0, \end{cases} \tag{4.2}$$

which become, after plugging in functions F in (3.23) and F_{x_0} in (3.24),

$$\begin{cases} (2v_+ N(x_0) - v_-)(\bar{x} - x_0) - (2u_+ N(x_0) - u_-)(\bar{y} - \phi(x_0)) = 0, \\ 2\frac{J\phi''(x_0)}{n_+^2(x_0)}\left(v_+(\bar{x} - x_0) - u_+(\bar{y} - \phi(x_0))\right) - n_-(x_0) = 0, \end{cases} \tag{4.3}$$

where

$$\bar{x} := x - u_- t, \qquad \bar{y} := y - v_- t.$$

Taking x_0 as a parameter, then (4.3) gives a unique solution for $(\bar{x}, \bar{y})$:

$$\begin{cases} \bar{x} = \bar{x}(x_0) = x_0 + \frac{n_-(x_0)n_+(x_0)}{2J^2\phi''(x_0)}\big(- u_+u_-\phi''(x_0) + 2u_+v_- - u_-v_+\big), \\ \bar{y} = \bar{y}(x_0) = \phi(x_0) + \frac{n_-(x_0)n_+(x_0)}{2J\phi''(x_0)}\big((u_+v_- - 2u_-v_+)\phi'(x_0) + v_+v_-\big). \end{cases} \tag{4.4}$$

Then we obtain the equations for the envelope surface in the parametric form:

$$\begin{cases} x = u_- t + \bar{x}(x_0), \\ y = v_- t + \bar{y}(x_0), \end{cases} \tag{4.5}$$

where x_0 and t are parameters. We note that, taking $t = 0$ in (4.5), we obtain the equation in the parametric form for the intersection curve of the envelope surface with the plane $t = 0$:

$$\begin{cases} x = \bar{x}(x_0), \\ y = \bar{y}(x_0), \end{cases} \tag{4.6}$$

which solves uniquely the following equations:

$$\begin{cases} F(x,y,0,x_0)=0, \\ F_{x_0}(x,y,0,x_0)=0. \end{cases} \tag{4.7}$$

We note that envelope surface (4.5) is generated from curve (4.6) as its directrix and $(u_-, v_-, 1)$ as the direction of its generator. Therefore, the shape of curve (4.6) yields the shape of the envelope surface. The cusp of the envelope surface is also the straight line which is parallel to direction $(u_-, v_-, 1)$.

4.2. Equations for the cusp. In order to analyze the shape of the envelope, it requires us to determine the cusp, which is governed by the following equations:

$$\begin{cases} F(x,y,t,x_0)=0, \\ F_{x_0}(x,y,t,x_0)=0, \\ F_{x_0x_0}(x,y,t,x_0)=0. \end{cases} \tag{4.8}$$

If we directly calculate $F_{x_0x_0}(x,y,t,x_0)$ from the definition of F, the formula is very complicated. Instead, we now find an equivalent equation of $F_{x_0x_0}(x,y,t,x_0)=0$. Denote

$$Q(x,y,t,x_0) := 2J\phi''(x_0)\left(v_+(x-u_-t-x_0)-u_+(y-v_-t-\phi(x_0))\right)-n_-(x_0)n_+^2(x_0).$$

Then

$$Q(x,y,t,x_0) = n_+^2(x_0)F_{x_0}. \tag{4.9}$$

We see that equation $F_{x_0}=0$ is equivalent to $Q(x_0)=0$. Moreover,

$$Q_{x_0} = (n_+^2(x_0))_{x_0}F_{x_0} + n_+^2(x_0)F_{x_0x_0} = n_+^2(x_0)F_{x_0x_0}.$$

Since $F_{x_0}=0$, then $F_{x_0x_0}=0$ is equivalent to equation $Q_{x_0}=0$. Using $F_{x_0}=0$, we can calculate Q_{x_0} to obtain

$$Q_{x_0} = \frac{n_+^2(x_0)}{\phi''(x_0)}\left(n_-(x_0)\phi'''(x_0)+3u_-\phi''(x_0)^2\right). \tag{4.10}$$

Obviously, equation $Q_{x_0}=0$ is equivalent to

$$G(x_0) := n_-(x_0)\phi'''(x_0)+3u_-\phi''(x_0)^2=0.$$

Thus, cusp equations (4.8) are equivalent to

$$\begin{cases} F(x,y,t,x_0)=0, \\ F_{x_0}(x,y,t,x_0)=0, \\ G(x_0)=0. \end{cases} \tag{4.11}$$

Since we assume that $G(x_0)=0$ has a unique solution $x_0=x^*$, then, after substituting $x_0=x^*$ into (4.11), we obtain the equations for single cusp $\widehat{PP_*}$:

$$\begin{cases} F(x,y,t,x^*)=0, \\ F_{x_0}(x,y,t,x^*)=0, \end{cases}$$

which are equivalent to the following equations:

$$
(4.12)\qquad \begin{cases} x = u_-t + \bar{x}(x^*) \\ \quad = u_-t + x^* + \frac{n_-(x^*)n_+(x^*)}{2J^2\phi''(x^*)}\left(-u_+u_-\phi''(x^*) + 2u_+v_- - u_-v_+\right), \\ y = v_-t + \bar{y}(x^*) \\ \quad = v_-t + \phi(x^*) + \frac{n_-(x^*)n_+(x^*)}{2J^2\phi''(x^*)}\left(\phi'(x^*)(u_+v_- - 2u_-v_+) + v_+v_-\right). \end{cases}
$$

4.3. Shape of the envelope surface. In order to investigate the shape of envelope surface (4.2), we need to analyze the shape of its directrix (4.6):

$$
x = \bar{x}(x_0), \qquad y = \bar{y}(x_0),
$$

which is a curve on the $x - y$ plane, with x_0 as a parameter. We now use parametric equation (4.6) to compute the first and second derivatives of directrix curve $y = y(x)$:

$$
\frac{dy}{dx} = \frac{\bar{y}_{x_0}}{\bar{x}_{x_0}}, \quad \frac{d^2y}{d^2x} = \frac{\left(\frac{\bar{y}_{x_0}}{\bar{x}_{x_0}}\right)_{x_0}}{\bar{x}_{x_0}}.
$$

It requires us to find first the two derivatives $\bar{x}_{x_0}$ and $\bar{y}_{x_0}$. We note that $x = \bar{x}(x_0)$ and $y = \bar{y}(x_0)$ satisfy (4.7), which is equivalent to the following equations by (4.9):

$$
(4.13)\qquad \begin{cases} F(x, y, 0, x_0) = 0, \\ Q(x, y, 0, x_0) = 0. \end{cases}
$$

Recall that

$$
\begin{aligned}
F(x, y, 0, x_0) &= (x - x_0)\left(2v_+N(x_0) - v_-\right) - (y - \phi(x_0))\left(2u_+N(x_0) - u_-\right), \\
Q(x, y, 0, x_0) &= 2J\phi''(x_0)\left(v_+(x - x_0) - u_+(y - \phi(x_0))\right) - n_-(x_0)n_+^2(x_0).
\end{aligned}
$$

Substituting $x = \bar{x}(x_0)$, $y = \bar{y}(x_0)$ in (4.13) and then taking the derivatives with respect to x_0 in the resulting equations, we obtain

$$
\begin{aligned}
\frac{dF(x, y, 0, x_0)}{dx_0} &= \left(2v_+N(x_0) - v_-\right)\bar{x}_{x_0} - (2u_+N(x_0) - u_-)\bar{y}_{x_0} + F_{x_0}(x, y, 0, x_0) = 0, \\
\frac{dQ(x, y, 0, x_0)}{dx_0} &= 2J\phi''(x_0)\left(v_+\bar{x}_{x_0} - u_+\bar{y}_{x_0}\right) + Q_{x_0}(x, y, 0, x_0) = 0.
\end{aligned}
$$

Since $F_{x_0}(x, y, 0, x_0) = 0$ from (4.7), we have the following equations for $\bar{x}_{x_0}$ and $\bar{y}_{x_0}$:

$$
\begin{cases} (2v_+N(x_0) - v_-)\bar{x}_{x_0} - (2u_+N(x_0) - u_-)\bar{y}_{x_0} = 0, \\ v_+\bar{x}_{x_0} - u_+\bar{y}_{x_0} = -\frac{1}{2J\phi''(x_0)}Q_{x_0}(x, y, 0, x_0), \end{cases}
$$

which have the unique solution

$$
(4.14)\qquad \begin{cases} \bar{x}_{x_0} = -\frac{1}{2J^2\phi''(x_0)}(2u_+N(x_0) - u_-)Q_{x_0}(x, y, 0, x_0), \\ \bar{y}_{x_0} = -\frac{1}{2J^2\phi''(x_0)}(2v_+N(x_0) - v_-)Q_{x_0}(x, y, 0, x_0). \end{cases}
$$

Then

$$\frac{dy}{dx} = \frac{\bar{y}_{x_0}}{\bar{x}_{x_0}} = \frac{2v_+N(x_0) - v_-}{2u_+N(x_0) - u_-}, \tag{4.15}$$

and

$$\left(\frac{\bar{y}_{x_0}}{\bar{x}_{x_0}}\right)_{x_0} = \left(\frac{2v_+N(x_0) - v_-}{2u_+N(x_0) - u_-}\right)_{x_0} = \frac{2J^2\phi''(x_0)}{(2u_+N(x_0) - u_-)^2 n_+^2(x_0)},$$

which gives

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{\left(\frac{\bar{y}_{x_0}}{\bar{x}_{x_0}}\right)_{x_0}}{\bar{x}_{x_0}} &= -\frac{4J^4(\phi''(x_0))^2}{(2u_+N(x_0) - u_-)^3 n_+^2(x_0) Q_{x_0}(x, y, 0, x_0)} \\ &= -\frac{4J^4(\phi''(x_0))^3}{(2u_+N(x_0) - u_-)^3 n_+^4(x_0) G(x_0)}. \end{aligned} \tag{4.16}$$

Denote

$$k(x_0) = \frac{2v_+N(x_0) - v_-}{2u_+N(x_0) - u_-}, \qquad M(x_0) = 2u_+N(x_0) - u_-.$$

Then

$$k_{x_0} = \frac{2J^2\phi''(x_0)}{n_+^2(x_0)(2u_+N(x_0) - u_-)^2} > 0$$

and

$$M_{x_0} = \frac{2u_+J\phi''(x_0)}{n_+^2(x_0)} \neq 0,$$

which show that k is strictly increasing and M is strictly monotone in x_0. Note that

$$M(x_0) = \frac{1}{n_+(x_0)}\big(- u_+u_-\phi'(x_0) + (2u_+v_- - u_-v_+)\big) = -\frac{H(x_0)}{n_+(x_0)},$$

where $H(x_0) = u_+u_-\phi'(x_0) - (2u_+v_- - u_-v_+)$. Since we assume that $H(x)$ has unique zero point x^{**}, then $M(x_0)$ has unique zero point x^{**}.

Lemma 4.1. $\lim_{x_0 \to x^{**}\pm} k(x_0) = \mp\infty$.

Proof. We know that

$$M(x^{**}) = 2u_+N(x^{**}) - u_- = 0.$$

We first show that x^{**} is not a zero point of $2v_+N(x_0) - v_-$. Otherwise, if

$$2v_+N(x^{**}) - v_- = 0,$$

then

$$J = \begin{vmatrix} u_+ & v_+ \\ u_- & v_- \end{vmatrix} = 0,$$

which is a contradiction. Thus, $2v_+N(x^{**}) - v_- \neq 0$. From the definition of $k(x_0)$, we see that $k(x_0) \to \infty$ or $-\infty$ as $x_0 \to x^{**}$. Suppose that $k(x_0) \to -\infty$ as $x_0 \to$

$x^{**}-$. Since $k(x_0)$ is increasing in x_0, then $k(x_0) = -\infty$ for all $x_0 < x^{**}$, which is impossible. Thus, $k(x_0) \to \infty$ as $x_0 \to x^{**}-$, which implies that $k(x_0) \to -\infty$ as $x_0 \to x^{**}+$. □

LEMMA 4.2. $(x_0 - x^{**})M(x_0) < 0$ *for any* $x_0 \neq x^{**}$.

Proof. We first notice that

$$\begin{aligned}&(2u_+N(x_0) - u_-, 2v_+N(x_0) - v_-) \cdot (-\phi'(x_0), 1)\\ &= -\phi'(x_0)M(x_0) + (2v_+N(x_0) - v_-) = 2n_+(x_0)N(x_0) - n_-(x_0) = n_-(x_0) > 0\end{aligned}$$

for any $x_0 \in \mathbb{R}$. Then, from $M(x^{**}) = 0$, we have

$$2v_+N(x^{**}) - v_- = n_-(x^{**}) > 0.$$

From the continuity of $2v_+N(x_0) - v_-$ and $k(x_0)$ near $x_0 < x^{**}$, there is an interval $I = (x^{**} - \delta, x^{**})$ for some $\delta > 0$ such that, for any $x_0 \in I$,

$$2v_+N(x_0) - v_- > 0, \quad k(x_0) > 0,$$

and then

$$M(x_0) = 2u_+N(x_0) - u_- > 0.$$

Since $M(x_0)$ has unique zero point x^{**}, then $M(x_0) > 0$ for any $x_0 < x^{**}$. Similarly, we conclude that $M(x_0) < 0$ for any $x_0 > x^{**}$. □

Recall that x^* and x^{**} are the unique zero of the functions

$$G(x_0) = n_-(x_0)\phi'''(x_0) + 3u_-(\phi''(x_0))^2$$

and $M(x_0) = 2u_+N(x_0) - u_-$, respectively. Without loss of generality, we assume $x^* < x^{**}$. Since $G'(x^*) < 0$, then $G(x_0) > 0$ for $x_0 < x^*$ and $G(x_0) < 0$ for $x_0 > x^*$. Thus, if $x_0 \in (-\infty, x^*)$, then $M(x_0) > 0$, $G(x_0) > 0$, and

$$\frac{d^2y}{dx^2} = -\frac{4J^4(\phi''(x_0))^3}{n_+^4(x_0)M^3(x_0)G(x_0)} < 0;$$

if $x_0 \in (x^*, x^{**})$, then $M(x_0) > 0$, $G(x_0) < 0$, and

$$\frac{d^2y}{dx^2} = -\frac{4J^4(\phi''(x_0))^3}{n_+^4(x_0)M^3(x_0)G(x_0)} > 0;$$

and if $x_0 \in (x^{**}, +\infty)$, then $M(x_0) < 0$, $G(x_0) < 0$, and

$$\frac{d^2y}{dx^2} = -\frac{4J^4(\phi''(x_0))^3}{n_+^4(x_0)M^3(x_0)G(x_0)} < 0.$$

Hence, the graph of envelope surface $x = \bar{x}(x_0)$, $y = \bar{y}(x_0)$ looks like Figure 5. A shock surface inside the envelope surface, called an inner shock surface, will appear, which will be discussed in section 5.

5. Estimate of the inner shock surface. Let

$$S^*(x, y, t) := y - v_-t - \gamma(x - u_-t) \tag{5.1}$$

be shock surface S^* that is generated by the interaction between the two parts of intermediate states

$$(u_+N(x_i), v_+N(x_i)), \quad i = 1, 2,$$

where $x_1 = x_1(x, y, t)$ is the unique global implicit function determined by

$$\begin{cases} (2v_+N(x_1) - v_-)(x - x_1 - u_-t) - (2u_+N(x_1) - u_-)(y - \phi(x_1) - v_-t) = 0, \\ (2u_+N(x_1) - u_-)(x - x_1 - u_-t) + (2v_+N(x_1) - v_-)(y - \phi(x_1) - v_-t) > 0, \\ x_1 < x^*, \end{cases} \tag{5.2}$$

and $x_2 = x_2(x, y, t)$ is the unique global implicit function determined by

$$\begin{cases} (2v_+N(x_2) - v_-)(x - x_2 - u_-t) - (2u_+N(x_2) - u_-)(y - \phi(x_2) - v_-t) = 0, \\ (2u_+N(x_2) - u_-)(x - x_2 - u_-t) + (2v_+N(x_2) - v_-)(y - \phi(x_2) - v_-t) > 0, \\ x_2 > x^*. \end{cases} \tag{5.3}$$

The reason why we can set inner shock surface S^* as form (5.1) is that all the contour surfaces of $(u_+N(x_i), v_+N(x_i))$, $i = 1, 2$, are the planes parallel to direction $(u_-, v_-, 1)$. Thus, S^* has the generator parallel to $(u_-, v_-, 1)$ and passes through the cusp of envelope surface Π. The cusp itself is also a ray parallel to $(u_-, v_-, 1)$ and passes through starting point $P^* : (x_p, y_p)$ at $t = 0$, i.e.,

$$y_p = \gamma(x_p),$$

where (x_p, y_p) is the unique solution of

$$\begin{cases} F(x_p, y_p, 0, x^*) = 0, \\ F_{x_0}(x_p, y_p, 0, x^*) = 0. \end{cases} \tag{5.4}$$

Note that

$$S_t^* = -v_- + \gamma'(x - u_-t)u_-, \quad S_x^* = -\gamma'(x - u_-t), \quad S_y^* = 1.$$

Then, jump condition for shock S^* is as follows:

$$S_t^* + u_+(N(x_1) + N(x_2))S_x^* + v_+(N(x_1) + N(x_2))S_y^* = 0$$

becomes

$$\begin{cases} -v_- + \gamma'(x - u_-t)u_- - u_+W\gamma'(x - u_-t) + v_+W = 0, \\ \gamma(x_p) = y_p, \end{cases} \tag{5.5}$$

where $W = N(x_1) + N(x_2)$. Let $\alpha = x - u_-t$. Then $y - v_-t = \gamma(\alpha)$ on S^*. The first equations in (5.2)–(5.3) become

$$(2v_+N(x_i) - v_-)(\alpha - x_i) - (2u_+N(x_i) - u_-)(\gamma(\alpha) - \phi(x_i)) = 0, \qquad i = 1, 2.$$

Therefore, x_i can be considered as a function of α and $\gamma(\alpha)$:

$$x_i = x_i(\alpha, \gamma(\alpha)).$$

Then (5.5) can be rewritten as the following Cauchy problem:

$$\begin{cases} \gamma'(\alpha) = \frac{v_+ W - v_-}{u_+ W - u_-}, \\ \gamma(x_p) = y_p. \end{cases} \tag{5.6}$$

Once we solve $\gamma(\alpha)$ from (5.6), then

$$S^*(x, y, t) = y - v_- t - \gamma(\alpha) = y - v_- t - \gamma(x - u_- t)$$

is the shock surface generated by the interaction between the two intermediate states.

Denote by Ω the region bounded by two branches Π_1 (for $x_0 < x^*$) and Π_2 (for $x_0 > x^*$) of envelope surface Π, where Π_1 and Π_2 are governed by the following equations:

$$\Pi_1 : \begin{cases} x = u_- t + \bar{x}(x_0), \\ y = v_- t + \bar{y}(x_0), \\ x_0 < x^*, \end{cases} \qquad \Pi_2 : \begin{cases} x = u_- t + \bar{x}(x_0), \\ y = v_- t + \bar{y}(x_0), \\ x_0 > x^*, \end{cases}$$

where $\bar{x}(x_0)$ and $\bar{y}(x_0)$ are given in (4.4) and $x_0 = x - u_- t$. We now prove that shock surface S^* lies inside region Ω, that is, S^* cannot escape region Ω.

Suppose that S^* would escape Ω. Without loss of generality, assume that S^* would intersect Π_1. Since both Π_1 and S^* are the surfaces with generators parallel to vector $(u_-, v_-, 1)$, we can write the equation of Π_1 in the form

$$y - v_- t = \Pi_1(x - u_- t), \quad x - u_- t < x^*.$$

Then, the intersection of S^* and Π_1 is determined by the equations

$$\begin{cases} y - v_- t = \gamma(x - u_- t), \\ y - v_- t = \Pi_1(x - u_- t). \end{cases} \tag{5.7}$$

Let $\alpha = x - u_- t$. Since $\gamma(\alpha)$ and $\Pi_1(\alpha)$ are different functions, the first point of the intersection of S^* and Π_1 yields that there exists some α^* such that $\gamma(\alpha^*) = \Pi_1(\alpha^*)$ and, for any α between α^* and x_p, one has

$$\gamma(\alpha) \neq \Pi_1(\alpha), \quad \text{i.e.,} \quad \gamma(\alpha) > \Pi_1(\alpha).$$

Then

$$\gamma'(\alpha^*) - \Pi_1'(\alpha^*) = \lim_{\alpha \to \alpha^* -} \frac{\gamma(\alpha) - \Pi_1(\alpha)}{\alpha - \alpha^*} \leq 0,$$

i.e.,

$$\gamma'(\alpha^*) \leq \Pi_1'(\alpha^*). \tag{5.8}$$

From (5.6), we have

$$\gamma'(\alpha^*) = \frac{v_+(N(x_1) + N(x_2)) - v_-}{u_+(N(x_1) + N(x_2)) - u_-}$$

and from (4.15), we get

$$\Pi_1'(\alpha^*) = \frac{2v_+ N(x_1) - v_-}{2u_+ N(x_1) - u_-},$$

where x_1, x_2 satisfy $x_1 < x^* < x_2$. For fixed x_1, let

$$I(x_0) := \frac{v_+(N(x_1) + N(x_0)) - v_-}{u_+(N(x_1) + N(x_0)) - u_-}.$$

Then

$$I'(x_0) = \frac{J^2 N(x_0)\phi''(x_0)}{(u_+(N(x_1) + N(x_0)) - u_-)^2} > 0.$$

Thus, $I(x_2) > I(x_1)$, i.e.,

$$\gamma'(\alpha^*) = I(x_2) > I(x_1) = \Pi_1'(\alpha^*),$$

which contradicts with (5.8). Therefore, S^* can not escape Ω.

6. Shock surfaces connecting U_m with U_+ in short time. By (3.4), shock surface $S(x, y, t) = 0$ connecting $U_m = (u_m, v_m)$, with $U_+ = (u_+, v_+)$ in short time satisfies

$$S_t + (u_m + u_+)S_x + (v_m + v_+)S_y = 0, \qquad S(x, y, 0) = y - \phi(x),$$

where $(u_m, v_m) = (u_+N(x_i), v_+N(x_i))$, $i = 1, 2$, and

$$x_1 = x_1(x, y, t) < x^*, \qquad x_2 = x_2(x, y, t) > x^*,$$

are determined by (5.2) and (5.3), respectively. We denote the two branches of shock surface $S(x, y, t) = 0$ by S_1 and S_2. Equation $S(x, y, t) = 0$ can be also expressed by the parametric form as

$$\begin{cases} x = x(\beta, t), \\ y = y(\beta, t), \\ t = t, \end{cases}$$

where $\beta \in \mathbb{R}$, $t \geqslant 0$ and $x(\beta, t), y(\beta, t)$ are the unique solutions of the following Cauchy problem:

$$\begin{cases} \frac{dx}{dt} = u_+N(x_i) + u_+, \\ \frac{dy}{dt} = v_+N(x_i) + v_+, \\ x|_{t=0} = \beta, \\ y|_{t=0} = \phi(\beta). \end{cases}$$

When time increases, shock $S = S_1 \cup S_2$ will intersect with inner shock S^*, which will be discussed in section 7.

7. Interaction between inner shock S^* and shock S. Recall that the equation of inner shock surface S^* is

$$S^* : y - v_-t = \gamma(x - u_-t),$$

where $\gamma(x)$ is the solution determined by Cauchy problem (5.6). Shock surface $S(x, y, t) = 0$ connecting intermediate state U_m with right state U_+ is discussed in section 6. In this section, we prove that S must intersect with S^*.

We first prove that surface Σ:

$$\Sigma(x,y,t) := y - 2v_+t - \phi(x - 2u_+t) = 0, \tag{7.1}$$

will intersect with S^*. If we prove that every generator of S^* intersects with Σ, then Σ must intersect with S^*. The following easy lemma will be useful.

LEMMA 7.1. *If there exists some $x^0 \in \mathbb{R}$ such that (x, y) satisfies*

$$y - \phi(x^0) - \phi'(x^0)(x - x^0) < 0,$$

with $\phi'' > 0$, then (x, y) also satisfies $y - \phi(x) < 0$.

Proof. Condition $y - \phi(x^0) - \phi'(x^0)(x - x^0) < 0$ implies that point (x, y) lies below the tangent line of the graph of function $\phi(x)$ at $x = x^0$. Since ϕ is convex, then its tangent line always lies below the graph of $\phi(x)$. Thus, point (x, y) lies below the graph, that is, $y < \phi(x)$. ∎

Express the arbitrary generator of S^* as

$$\ell : \begin{cases} x = u_-t + \alpha, \\ y = v_-t + \gamma(\alpha), \end{cases}$$

with parameters $t \geq 0$ and $\alpha \geq x^*$. When $t = 0$, the starting point of ℓ is $(\alpha, \gamma(\alpha))$ that is located in the region of $y - \phi(x) > 0$. Note that, for some x^0,

$$n_-(x^0) < 2n_+(x^0),$$

that is,

$$-\phi'(x^0)(u_- - 2u_+) + (v_- - 2v_+) < 0.$$

Thus, if t is large enough, one has

$$\begin{aligned} &(\gamma(\alpha) + v_-t - 2v_+t) - \phi(x^0) - \phi'(x^0)\left(\alpha + u_-t - 2u_+t - x^0\right) \\ &= \big(n_-(x^0) - 2n_+(x^0)\big)t + \big(\gamma(\alpha) - \phi(x^0) - \phi'(x^0)\alpha + \phi'(x^0)x^0\big) < 0. \end{aligned}$$

According to Lemma 7.1, we have

$$\gamma(\alpha) + v_-t - 2v_+t - \phi(\alpha + u_-t - 2u_+t) < 0,$$

which means that, if t is large enough, point $(\alpha + u_-t, \gamma(\alpha) + v_-t)$ on generator ℓ will be on the side of $\Sigma(x, y, t) < 0$, i.e.,

$$y - 2v_+t - \phi(x - 2u_+t) < 0.$$

Thus, there must be a point on ℓ which is also on Σ, that is, ℓ intersects with Σ.

Along direction $(u_+, v_+, 0)$, the slope of the curve of S with projection parallel to ℓ is

$$\begin{aligned} &\frac{u_+}{\sqrt{u_+^2 + v_+^2}}\frac{dx}{dt} + \frac{v_+}{\sqrt{u_+^2 + v_+^2}}\frac{dy}{dt} \\ &= \frac{u_+}{\sqrt{u_+^2 + v_+^2}}u_+(N(x_0) + 1) + \frac{v_+}{\sqrt{u_+^2 + v_+^2}}v_+(N(x_0) + 1) \\ &= \sqrt{u_+^2 + v_+^2}(N(x_0) + 1) \\ &> 2\sqrt{u_+^2 + v_+^2}\ . \end{aligned}$$

However, the slope of the curve on Σ with projection parallel to ℓ is

$$\frac{u_+}{\sqrt{u_+^2+v_+^2}}\cdot 2u_+ + \frac{v_+}{\sqrt{u_+^2+v_+^2}}\cdot 2v_+ = 2\sqrt{u_+^2+v_+^2} < \frac{u_+}{\sqrt{u_+^2+v_+^2}}\frac{dx}{dt} + \frac{v_+}{\sqrt{u_+^2+v_+^2}}\frac{dy}{dt}\ .$$

Thus, S is located in the region of $\Sigma(x,y,t)>0$, i.e., S is in the region of

$$y-2v_+t-\phi(x-2u_+t)>0,$$

which means that all points (x,y,t) with $\Sigma(x,y,t)<0$ satisfy $S(x,y,t)<0$.

Notice that starting point $(\alpha,\gamma(\alpha))$ of ℓ is in region $S(x,y,t)>0$. Since we have proved that, if t is big enough, there exists a point (x',y',t) on ℓ such that $\Sigma(x',y',t)<0$, then this point (x',y',t) is also in the area where ℓ intersects with $S(x,y,t)=0$. Since ℓ is an arbitrary generator of inner shock surface S^* and the above procedure and results are true for all generators, then S^* must intersect with $S(x,y,t)=0$.

8. Formation of triple-shock pattern. Recall that inner shock surface S^*, starting from the cusp curve with the following equation:

(8.1)
$$\begin{cases} x = u_-t+\bar{x}(x^*) = u_-t + x^* + \frac{n_-(x^*)n_+(x^*)}{2J^2\phi''(x^*)}(-u_+u_-\phi''(x^*)+2u_+v_- - u_-v_+), \\ y = v_-t+\bar{y}(x^*) = v_-t+\phi(x^*) + \frac{n_-(x^*)n_+(x^*)}{2J^2\phi''(x^*)}(\phi'(x^*)(u_+v_- - 2u_-v_+)+v_+v_-), \end{cases}$$

is governed by

$$S^*:\ y-v_-t=\gamma(x-u_-t).$$

Thus, $y=\gamma(\alpha)$ is the directrix of S^*, and $\{u_-,v_-,1\}$ is the direction of the generators of S^*.

From (5.6), the slope of the tangent line of directrix $y=\gamma(\alpha)$ is

$$\gamma'(\alpha)=\frac{v_+W-v_-}{u_+W-u_-},$$

where $W=N(x_1)+N(x_2)$ and x_i, $i=1,2$, can be expressed as functions of α. Therefore, the direction vector of the tangent line is

$$\vec{\tau}:=(u_+W-u_-, v_+W-v_-, 0).$$

On the other hand, shock surface S is governed by following parametric equations:

$$x=x(\beta,t),\quad y=y(\beta,t), \tag{8.2}$$

where $\beta\in(-\infty,\infty)$, $t\geqslant 0$, and $x(\beta,t),y(\beta,t)$ are the solutions of the following Cauchy problem:

$$\begin{cases} \frac{dx}{dt}=u_+N(x_0)+u_+, \\ \frac{dy}{dt}=v_+N(x_0)+v_+, \\ (x,y)|_{t=0}=(\beta,\phi(\beta)). \end{cases} \tag{8.3}$$

Since

$$\frac{dy}{dx} = \frac{u_+}{v_+},$$

then, for any given fixed β, curve (8.2) is the curve on S whose projection is the straight line that is parallel to vector

$$\vec{\theta} := (u_+, v_+, 0).$$

Now

$$\vec{\tau} \times \vec{\theta} = (0, 0, J).$$

In addition, the direction of the characteristic line that connects with singular point $(\bar{x}(x^*), \bar{y}(x^*))$ is

$$\vec{\eta} := (2u_+N(x^*) - u_-, 2v_+N(x^*) - v_-, 0)$$

and then

$$\vec{\eta} \times \vec{\theta} = (0, 0, J).$$

Since, along the normal direction of curve $y - \phi(x) = 0$,

$$(-\phi'(x), 1) \cdot (u_+, v_+) = n_+(x) > 0,$$

then $\vec{\theta} = (u_+, v_+, 0)$ always points to the side of $y - \phi(x) > 0$. Thus, there are two cases of the triple-shock pattern depending on the sign of J.

Case 1. If

$$J = \begin{vmatrix} u_+ & v_+ \\ u_- & v_- \end{vmatrix} > 0,$$

then inner shock S^* intersects with S_2. Here S_2 is the shock surface between the part of intermediate states $(u_+N(x_2), v_+N(x_2))$ and (u_+, v_+) and can be expressed in the parametric form:

$$S_2: \qquad x = x(\beta, t), \qquad y = y(\beta, t), \qquad \beta > x^*,\ t \geq 0, \tag{8.4}$$

where $x(\beta, t)$ and $y(\beta, t)$ are given in Remark 3.2 as well as in (8.2) and (8.3) and $x_2 = x_2(x, y, t)$ is given in (5.3). The intersection point of cusp (8.1) and S_2 is

$$Q_1 := (\bar{x}(x^*) + u_-T_1,\ \bar{y}(x^*) + v_-T_1,\ T_1)$$

for some time $T_1 > 0$. Let Γ_2^* be the intersection curve of S^* and S_2. Note that Q_1 is the lowest starting point of Γ_2^*. Let S_1 be the shock surface connecting the other part of intermediate states $(u_+N(x_1), v_+N(x_1))$ and (u_+, v_+), with the parametric form:

$$S_1: \qquad x = x(\beta, t), \qquad y = y(\beta, t), \qquad \beta < x^*,\ t \geq 0, \tag{8.5}$$

where $x(\beta, t)$ and $y(\beta, t)$ are the same as those in (8.2) and $x_1 = x_1(x, y, t)$ is given in (5.2). Set curve C_1 to be

$$C_1: \qquad x = x(\beta, T_1), \quad y = y(\beta, T_1), \quad \beta < x^*,$$

which is the curve of intersection between S_1 and plane $t = T_1$. Thus, after crossing $\Gamma_2^* \cup C_1$ when $t > T_1$, there exists a shock surface $S_1^{\Gamma_2^*}$ connecting intermediate states $(u_+N(x_1), v_+N(x_1))$ and (u_+, v_+), which also passes Γ_2^*. Note that $S_1^{\Gamma_2^*}$ appears only when $t > T$. Denote $S_1^{c_1}$ the shock surface between state $(u_+N(x_1), v_+N(x_1))$ and (u_+, v_+), which passes curve C_1. Thus, in this case, the triple shock surfaces are S^*, S_2, and $S_1^{\Gamma_2^*}$, and Γ_2^* is the common curve where these three curves intersect together. Such a structure only appears when $t > T_1$.

Case 2. If

$$J = \begin{vmatrix} u_+ & v_+ \\ u_- & v_- \end{vmatrix} < 0,$$

then S^* intersects with S_1, where S_1 is the shock surface connecting the part of intermediate states $(u_+N(x_1), v_+N(x_1))$ and (u_+, v_+), with the parametric form of (8.5). The intersection point of cusp (8.1) and S_1 is

$$Q_2 := (\bar{x}(x^*) + u_-T_2,\ \bar{y}(x^*) + v_-T_2,\ T_2)$$

for some time $T_2 > 0$. Let Γ_1^* be the intersection curve between S^* and S_1. Note that Q_2 is the lowest starting point of Γ_1^*. Set curve C_2 to be

$$C_2: \qquad x = x(\beta, T_2), \quad y = y(\beta, T_2), \quad \beta > x^*,$$

which is the curve of intersection between S_2 and plane $t = T_2$. After crossing $\Gamma_1^* \cup C_2$ when $t > T_2$, there exists a shock surface $S_2^{\Gamma_1^*}$ between intermediate states $(u_+N(x_2), v_+N(x_2))$ and (u_+, v_+), which also passes Γ_1^*. Note that $S_2^{\Gamma_1^*}$ appears only when $t > T_2$. Denote $S_2^{c_2}$ the shock surface between intermediate states $(u_+N(x_2), v_+N(x_2))$ and (u_+, v_+), which passes curve C_2. Thus, in this case, the triple shock surfaces are S^*, S_1, and $S_2^{\Gamma_1^*}$, and Γ_1^* is the common curve which these three curves intersect together. Such a structure only appears when $t > T_2$.

Thus, the proof of Theorem 2.1 is complete.

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