THE EIGENCURVE AT EISENSTEIN WEIGHT ONE POINTS

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A talk in the Oxford Number Theory Seminar.
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1. \( p \)-adic families of Eisenstein families

Let \( k \geq 4 \) be even, and let \( p \) be an odd prime. We have the classical Eisenstein series of weight \( k \) and level 1,
\[
E_k(q) = \frac{\zeta(1-k)}{2} + \sum_{n>0} \left( \sum_{d|n} d^{k-1} \right) q^n \in M_k(1, \mathbb{Q}).
\]
The constant terms of the Eisenstein series satisfy the Kummer congruences: for \( k, h \in \mathbb{Z} \), \( k \equiv h \pmod{(p-1)p^{a}} \), we have
\[
\zeta(1-k)(1-p^k) \equiv \zeta(1-h)(1-p^h) \pmod{p^{a+1}}.
\]
In addition, we have the obvious congruences
\[
p \nmid d \implies d^{k-1} \equiv d^{h-1} \pmod{p^{a+1}}.
\]
Let
\[
E_{k, \text{ord}}(q) = E_k(q) - p^{k-1}E_k(q^p)
\]
\[
= (1 - p^{k-1}) \frac{\zeta(1-k)}{2} + \sum_{n>0} \left( \sum_{d|n,l|n} d^{k-1} \right) q^n \in M_k(p, \mathbb{Q})^{\text{ord}}.
\]

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For an example of such characters, consider that given a pair \((\phi, k)\) where 
\[ \phi : (\mathbb{Z}/Np^a\mathbb{Z})^\times \rightarrow \mathbb{C}_p^\times, \quad k \in \mathbb{Z}_{\geq 1}, \]
there is a character 
\[ \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \ni (x, y) \mapsto \phi(\bar{x}, \bar{y}) \cdot x^k. \]
These characters are exactly the \textit{classical weights}, by definition.

**Theorem 1.3** (Serre). Let \( N = 1; \) there exists a \( \Lambda \)-adic \( q \)-series 
\[ \mathcal{E}_1(q) = \sum_{n \geq 1} a_n q^n \in K(\Lambda) \oplus q\Lambda[q] \]
such that its weight \( k \)-specialization is 
\[ \mathcal{E}_1 \otimes_{\Lambda, (1, k)} \mathbb{C}_p = E_k^{\text{cusp}} \]
for \( k \geq 4. \)

Thus, \( \mathcal{E}_1 \) can be thought of as a family parametrized by the weight space.

1.1. \textbf{Classical weight one phenomena}. We need to introduce an odd character in order to get an Eisenstein series of weight one. Let 
\[ \phi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \text{ primitive, } (p, N) = 1. \]
Let 
\[ E_1(1, \phi)(q) = \frac{L(\phi, 0)}{2} + \sum_{n \geq 1} \left( \sum_{d|n} \phi(d)d^0 \right) q^n \]
\[ = \frac{L(\phi, 0)}{2} + \sum_{n \geq 1} \left( \sum_{d|n} \phi(n/d)d^0 \right) q^n = E(\phi, 1)(q). \]
The \( p \)-th Hecke polynomial is 
\[ X^2 - a_p(E_1(1, \phi)) + \phi(p) = (X - 1)(X - \phi(p)). \]
So we have the two stabilizations 
\[ f_1 = E_1(1, \phi)(q) - \phi(p)E_1(1, \phi)(q^p), \text{ on which } U_p = 1, \]
which interpolates to \( \mathcal{E}_{1, \phi} \), and 
\[ f_1 = E_1(1, \phi)(q) - E_1(1, \phi)(q^p), \text{ on which } U_p = \phi(p), \]
interpolating to \( \mathcal{E}_{\phi, 1} \). Therefore, we have two possibilities:

**Case 1:** \( \phi(p) \neq 1 \), and therefore there the two Eisenstein families \( \mathcal{E}_{1, \phi} \) and \( \mathcal{E}_{\phi, 1} \) do not meet in weight 1.

**Case 2:** we have \( f = f_1 = f_2 \), and the two Eisenstein families meet in weight 1. In this case, we find that 
\[ a_0(f) = (1 - \phi(p))L(\phi, 0) = 0. \]
The constant term is given by the associated \( p \)-adic \( L \)-function \( L_p(\phi\omega, s) \) evaluated at the trivial zero \( s = 0 \).

**Fact.** \( f \) is a cuspidal form when viewed as a \( p \)-adic form.
Goal. Understand all cuspidal families passing through $f$, including how it lies over the weight space via the weight map $w$.

Here $C_N$ denotes the eigencurve of tame level $N$, and $C_N^{\text{cusp}}$ is its closed cuspidal sublocus. We have that $f \in C_N(\mathbb{C}_p)$. We remark that $C_N(\mathbb{C}_p)$ is naturally isomorphic to the set of systems of Hecke eigenvalues of overconvergent forms of tame level $N$ and of finite slope.

2. Definition of a cuspidal deformation ring

Let $\Lambda(1)$ denote the completed local ring of $W$ at 1. This is isomorphic to $E[[X]]$, where $E/\mathbb{Q}_p$ is generated by the values of $\phi$.

Heuristically, we expect that $\Lambda(1)$ = $E[[X]]$-families of cusp forms deforming $f$ biject with representations $\rho: G_\mathbb{Q} \to \text{GL}_2(E[[X]])$ such that $(\rho \mod X) = 1 \oplus \phi$, and $\rho$ is generically irreducible.

This behavior was studied by Ribet.

Lemma 2.1 (Ribet). Let $A$ be a complete DVR with $A/\pi A \cong k$. Let $K = \text{Frac}(A)$. If $\rho: G \to \text{GL}_K(V)$ is two-dimensional and irreducible, while $\bar{\rho} := \rho (\mod \pi)$ satisfies $\bar{\rho}^{ss} = \chi_1 \oplus \chi_2$, then there exists $L \subseteq V$, a $G$-stable $A$-lattice, such that $L \otimes_A k$ determines a non-trivial extension $\text{Ext}(\chi_1, \chi_2)$.

In this situation, write $\rho^u := \begin{pmatrix} \phi & * \\ 0 & 1 \end{pmatrix}$, (where $u$ stands for “upper triangular”) and write $R^\text{ord}_{\rho^u}$ for the deformation ring for $\rho^u$. This sends a completely Noetherian local $E$-algebra with residue field $E$ to the set of $\rho^u_A: G_\mathbb{Q} \to \text{GL}_2(A)$ such that

- $(\rho^u_A \mod m_A) = \rho^u$
- $\rho^u_A$ is ordinary, i.e. $\rho^u_A|_{G_{\mathbb{Q}_p}} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

with an unramified quotient.

We can calculate that the tangent space $t_{R^\text{ord}_{\rho^u}} := \text{Hom}(R^\text{ord}_{\rho^u}, E[\varepsilon]/\varepsilon^2)$ is a 2-dimensional $E$-vector space. This does not show that there is a unique cuspidal locus – this would follow from being 1-dimensional. This is not hard to
see – we can write down a deformation to \( E[X] \) that is not cuspidal. Namely, \( \mathcal{E}_{1,\phi} \) gives rise to
\[
\begin{pmatrix}
\phi \chi & * \\
0 & 1
\end{pmatrix}
\]
where \( \chi \) is the log-cyclotomic deformation specializing to the trivial character at \( X = 0 \), and where we simply choose an extension class \( * \) that, going modulo \( X \), is equal to that of \( \rho^u \).

To eliminate this, we produce \( R_{\text{ord}} \), where \( l \) is for “lower triangular,” similarly to before. The ordinary flag is lower triangular.

Thus we set up \( R_{\text{cusp}} \), a deformation ring classifying pairs, one lower triangular and one upper triangular deformation as above,
\[
(\rho^u_A, \rho^l_A),
\]
such that
\[
\text{Tr} \rho^u_A = \text{Tr} \rho^l_A, \quad \text{and} \quad \text{Tr} \rho^u_{A,I_p} = \text{Tr} \rho^l_{A,I_p}.
\]
We prove that the tangent space of \( R_{\text{cusp}} \) is one-dimensional. Thus we have

**Theorem 2.2** (Betina–Dimitrov–P). *The cuspidal eigencurve is étale at \( f \).*

Describing the tangent space of the cuspidal deformation ring, one obtains the derivatives of the coefficients of the unique cuspidal family \( F \) specializing to \( f \) at weight 1. These coefficients satisfy a congruence with a linear combination of Eisenstein series that can be written in terms of \( \mathcal{L} \)-invariants. Let \( H \) be the splitting field of the character \( \phi \) and consider
\[
U_{\phi} \subset (\mathcal{O}_H[1/p]^\times \otimes E)^{\phi^{-1}}.
\]
It is 1-dimensional over \( E \). There exists a generator \( u_{\phi} \) so that \( \langle u_{\phi} \rangle = U_{\phi} \). The \( \mathcal{L} \)-invariant of \( \phi \) is
\[
\mathcal{L}(\phi) = -\frac{\log_v(u_{\phi})}{\text{ord}_v(u_{\phi})}.
\]

**Theorem 2.3.** Let \( F \in E[[X]][[q]] \) be the unique cuspidal form reducing to \( f \). Then
\[
a_{\ell}(F) = \frac{\mathcal{L}(\phi^{-1})}{\mathcal{L}(\phi) + \mathcal{L}(\phi^{-1})} a_{\ell}(\mathcal{E}_{1,\phi}) + \frac{\mathcal{L}(\phi)}{\mathcal{L}(\phi) + \mathcal{L}(\phi^{-1})} a_{\ell}(\mathcal{E}_{\phi,1}) \pmod{X^2}
\]
for primes \( \ell \neq p \).