The subtitle of this talk is “$S = T$.”

1. Function field geometric Langlands

Before we discuss the diagram of the last talk in some more detail, let us go back to V. Lafforgue’s global function field Langlands for some more motivation. Recall that this is all about the “automorphic to Galois” direction. That is, we want to produce unramified Langlands parameters

$$\sigma : \pi_1(X, \overline{\eta}) \rightarrow \hat{G}(\overline{F})$$

from automorphic representations. To do this, we use the space

$$C^{\text{cusp}}_c(\text{Bun}_G(F_q), \mathbb{Q}_\ell),$$

for which Lafforgue constructs an action by $B$, the algebra of excursion operators. As we saw in the previous talk, these come from “creation” $\otimes W$, “Galois action” $\cdot (\gamma_i)$, and annihilation. (See V. Lafforgue’s 2018 ICM talk for more on this.)

The excursion operators induce a decomposition into generalised eigenspaces

$$C^{\text{cusp}}_c(-) = \bigoplus_v H_v,$$

where $v$ ranges over $v : B \rightarrow \mathbb{Q}_\ell$, and the point is that there is for any $v$ a unique unramified Langlands parameter $\sigma$ such that

$$v(S_{f, \gamma_i}) = f(\sigma(\gamma_i)),$$

thus (implicitly) defining the unramified Langlands parameter. Once this is done, we want to relate this back to the Hecke algebra $\mathcal{H}_G$. More precisely, to describe $\sigma$ it would be useful if the following hold:

- $\mathcal{H}_G$ should respect $\bigoplus_v H_v$.
- We would like $T_{V,x}$ to act on $\chi_V$ by

$$(\ast) \quad \chi_V(\sigma_V(\text{Frob}_x)).$$

One of V. Lafforgue’s main theorems is “$S = T$.”

**Theorem 1.1.** Let $V$ be an irreducible representation of $\hat{G}$. Let $x$ be a place of $F$. Let $f : G\backslash G^2 / G \rightarrow \mathbb{Q}_\ell$ be given by $(g_1, g_2) \mapsto \chi_V(g_1 g_2^{-1}) = \text{Tr}(g_1 g_2^{-1}|V)$. Let $\text{Frob}_x$ be any Frobenius at $x$. Then

$$S_{\{1,2\}, f, \text{Frob}_x, 1} = T_{V,x}.$$
In particular, there is a natural map of algebras $\mathcal{H}_G \to B$, the decomposition $\bigoplus_v H_v$ is respected by $\mathcal{H}_G$, and we do indeed get ($*$).

**Remark 1.2.** The image of $\mathcal{H}_G \to B$ is the subalgebra generated by those excursion operators with $#I \leq 2$. In the context of global Langlands for function fields, we really need $#I > 2$ to get the decomposition (when $G \neq GL_n$), i.e. $B$ is in general larger than (the image of) $\mathcal{H}_G$. Therefore, the statement “$S = T$” usually refers to an equality of operators, not of algebras (i.e. we may in general have $\mathcal{H}_G \neq B$).

2. **The local $S = T$ theorem**

So far

$$\begin{array}{ccc}
\text{Rep}(\hat{G}) & \xrightarrow{\text{Sat}} & P(\text{Hecke}_k) \\
V \mapsto \tilde{V} & \Phi & \\
\text{Coh}^{\text{ht}}(\hat{G}\sigma) & \xrightarrow{S} & P^{\text{Corr}}(\text{Sht}_k)
\end{array}$$

Where is the excursion operator in this picture? The excursion operator is the image $S(f)$ of a morphism $f$ in $\text{Coh}^{\text{ht}}(\hat{G}\sigma)$. Roughly, one can use the data of $f$ for the creation, the action of Galois is given by the Frobenius $\sigma$, and annihilation is then straightforward. Let us say a bit more about this:

Where does the Galois action come from in this picture? The Galois group in our setting is $\text{Gal}(\bar{k}/k)$. We can think of the objects in $\text{Coh}^{\text{ht}}(\hat{G}\sigma)$ as being coherent sheaves on $[\hat{G}\sigma/\hat{G}]$, just as we think of $\text{Rep}\hat{G}$ as being $\text{Coh}[\bullet/\hat{G}]$. But this is the stack of unramified Langlands parameters! That means it parameterises morphisms $\text{Gal}(\bar{k}/k) \to \hat{G} = \hat{G} \times \text{Gal}(\bar{k}/k)$ such that composition with the projection $\hat{G} \times \text{Gal}(\bar{k}/k) \to \text{Gal}(\bar{k}/k)$ gives the identity on $\text{Gal}(\bar{k}/k)$. The upshot is that in pulling back $V \in \text{Coh}[\bullet/\hat{G}] \mapsto \tilde{V} \in \text{Coh}[\hat{G}\sigma/\hat{G}]$ we essentially add a Galois action.

For any $f \in \text{Mor}(\text{Coh}^{\text{ht}}(\hat{G}\sigma))$, we think of the cohomological correspondence $S(f)$ as being an excursion operator. The main step in the construction of $S$ was to build a map $S : \text{Mor}(\text{Coh}^{\text{ht}}(\hat{G}\sigma)) \to \text{Mor}(P^{\text{Corr}}(\text{Sht}^\text{loc}_k))$. The idea is roughly to use the data of $f \in \text{Mor}(\text{Coh}^{\text{ht}}(\hat{G}\sigma))$ to set up a “creation operator”. Then use the Galois action by $\text{Frob} = \sigma$ (our Galois group is small, so this is really the only interesting thing to use). To be precise, of course $S(f)$ isn’t really an operator, but a cohomological correspondence, so perhaps we really should call $S(f)$ an “excursion correspondence”. But we get from this an operator by restricting attention to $\text{End} \to \text{Mor}$ and looking at the induced map on cohomology. The construction of $S$ is thus in some sense an “upgrade” of the excursion operators to correspondences, and a generalisations to any morphism, not just endomorphisms.

In the definition of $S(f)$, we used $\boxtimes$ – this was adding a leg, so we used two legs. Now from our earlier remark, this means we have reason to expect $\mathcal{H}_G = B$. Indeed, this is what happens, as we shall now discuss:

In the key diagram, we have the following objects mapping to each other:
This proves that $S(\tilde{1}) = \delta$.

**Theorem 2.1** (Thm. 6.0.1(2) of XZ). The following diagram commutes:

$$
\begin{array}{ccc}
\text{End } \tilde{1} & \xrightarrow{S} & \text{End } \delta_1 \\
\downarrow & & \downarrow \\
\Gamma(\hat{G}\sigma/\hat{G}, \mathcal{O}) & \xrightarrow{\text{Sat}^{\text{classical}}} & \mathcal{H}_G \\
\end{array}
$$

where the downward arrows are equalities. The rightward arrows are isomorphisms, “$S = T$”, and send $f \mapsto S(f)$. In particular, we do have “$B = \mathcal{H}_G$” in this case.

The rest of the talk will be essentially about the proof of this Theorem, including a detour to some related results.

3. **Beginning of the proof of $S = T$**

The proof will use cycles on affine Deligne–Lusztig varieties – but before, we need to recall some results from Timo’s talk, namely how to go from geometric Satake to classical Satake. In his talk, there was an isomorphism

$$\text{Sat} : \text{Rep}^L \hat{G} \xrightarrow{\sim} P^0_{L+G}(\text{Gr})$$

(this is the geometric Satake over $k$ rather than over $\overline{k}$). Then we have maps

$$\text{Rep}^L \hat{G} \to \Gamma(\hat{G}\sigma/\hat{G}, \mathcal{O}), \quad V \mapsto \chi_V|_{\hat{G}\sigma}$$

where $\chi_V : L^G \to \mathbb{Q}_l$ is the trace of the representation, and

$$P^0_{L+G}(\text{Gr}) \to H_G = C(\mathcal{O}\setminus \text{Gr}(k) \to \overline{\mathbb{Q}_l})$$

from the Grothendieck functions-sheaves correspondence,

$$\mathcal{A} \mapsto (\text{Gr}(k) \ni x \mapsto \sum_i (-1)^i \text{Tr}(\phi_x|_{\mathbb{H}^i(\mathcal{A}_x)})).$$

where $\bar{x}$ is some geometric point over $x$. These fit into a commutative diagram

$$
\begin{array}{ccc}
\text{Rep}^L \hat{G} & \xrightarrow{\text{Sat}} & P^0_{L+G}(\text{Gr}) \\
V \mapsto \chi_V|_{\hat{G}\sigma} & \downarrow & \downarrow_{f,s.} \\
\Gamma(\hat{G}\phi/\hat{G}, \mathcal{O}) & \xrightarrow{\text{Sat}^{\text{cl}}} & \mathcal{H}_G \\
\end{array}
$$

with vertical arrows surjective. The classical Satake $\text{Sat}^{\text{cl}}$ is thus the unique morphism making this diagram commute.

**Remark 3.3.** There is a minor issue: in Timo’s talk, we used $[\hat{G}\sigma/\hat{G}]$ instead of $[\hat{G}\sigma'/\hat{G}]$ which we want to use now, but that can be resolved via

$$\hat{G}\phi \to \hat{G}\sigma, \quad g\phi \mapsto \sigma(g^{-1})\sigma.$$

To compare the two morphisms $\text{Sat}_\sigma$ and $\text{Sat}_\phi : \text{Rep}^L \hat{G} \to H_G$ obtained from using $\hat{G}\sigma$ and $\hat{G}\phi$ respectively, we need to switch $[V] \mapsto [V^*]$. 

In light of the comparison of classical and geometric Satake, the idea for proving \( S = T \) is the following: since the two maps \([3.1]\) and \([3.2]\) are surjective (call this, with the horizontal map being \( \text{Sat} : \text{Rep}^L G \rightarrow P^0_{L+G}(\text{Gr}) \), “diagram \( 2^\ast \)”), it suffices to prove that for \( V \in \text{Rep}^L G \), we have an equality

\[
S(\chi_{V|G_\sigma}) = \text{Sat}^{\ast}(\chi_{V|G_\sigma}) \overset{\text{by def}}{=} \left( x \mapsto \sum_i (-1)^i \text{Tr} \left( \phi_x \mid H^i(A_{\xi}) \right) \right).
\]

To make something of the form \( S(\cdot) \) look like the RHS, we use

**Lemma 3.4** (Braverman–Varshavsky’s unpublished trace formula). Let \( (X, F) \) be defined over \( \mathbb{F}_q \), and let \( F \in D^b_c(X) \). Consider the cohomological correspondence \( u = X \overset{\alpha}{\rightarrow} X = X \), which by Toby’s talk induces a correspondence \( u^\# = X \times X \overset{\alpha \times \text{id}}{\rightarrow} X \rightarrow \ast \). Also by Toby’s talk, we always have the cohomological correspondence \( \delta_F = * \leftarrow X \overset{\Delta}{\rightarrow} X \times X \) defined by \( \mathbb{Q}_F \rightarrow \Delta^!(\mathcal{F} \boxtimes \mathcal{F}) \).

Then \( u^\# \circ \delta_F \) is supported on \( X(\mathbb{F}_q) \), where it is given by \( x \mapsto \text{Tr}(\phi \mid F_x) \).

This is Lemma A.2.22 in [XZ] but beware, for some reason they write composition of correspondences the other way round (i.e. here and in other places, they seem to write \( f \circ g \) for the correspondence which is \( f \) on the left composed with \( g \) on the right, whereas we choose this to write this as \( g \circ f \), since this is how the composition goes on the level of sheaves).

At this point, all that we have to do is to get our correspondence into this form, and then compare the traces. In order for the traces to match up, we want to apply the Lemma to \( X = \text{Gr}_\nu \) for some cocharacter \( \nu \), and \( F = \text{Sat}(V_\nu) \). In particular, we ultimately need a correspondence of Grassmannians, but currently we have a correspondence of shtukas. In the next section we will see how to get from one to the others, and we will do so via a little detour into a different topic.

### 4. The relation to Borel–Moore homology classes

Recall from Toby’s talk, an example.

**Example 4.1.** Let \( X_1, X_2 \) be perfectly smooth algebraic spaces of pure dimension, and let \( d_1, d_2 \) be any numbers (not necessarily the dimensions). Let \( X_1 \leftarrow C \rightarrow X_2 \) be any correspondence. Then

\[
\text{Corr}^C((X_1, \overline{\mathbb{Q}}_F(d_1)), (X_2, \overline{\mathbb{Q}}_F(d_2))) = H_{D^b_c(C)}(\overline{\mathbb{Q}}_F(d_1), \omega(d_2-2\dim X_2)) = H^{BM}_{2\dim X_2+d_1-d_2}(C)
\]

Recall from Rebecca’s talk,

\[
\text{Corr}_{\text{Sh}^\nu_{\text{loc}}}(\text{Sh}_{\nu}^\ast(-), \delta_1), (\text{Sh}_{\nu}^\ast(-), \delta_1)) = H^{BM}_{0}(\text{Sh}_{\nu_{\text{loc}}}(k)) = C(G(\mathcal{O}/\mathfrak{m}^n)) = \mathcal{H}_G.
\]

Let \( \tau \in X_*(Z_G) \) central, and \( \mu \) minuscule. Then for

\[
Y := \text{Sh}^\nu_{\tau_{\mu}}
\]

from James’s talk we have

\[
Y = [L^m_{\mu}(G) \backslash X_{\mu^*,\pi^*}],
\]
By the Lemma we have:
\[
\text{Corr}_Y((\text{Sht}_\tau, S(\tilde{V}_\tau)), (\text{Sht}_\mu, S(\tilde{V}_\mu))) = H_{(2p, \mu)}^{BM}(Y) = H_{(2p, \mu)}^{BM}(X_{\mu, v}(\tau))^G(\mathcal{O}/\mathcal{O}^\nu) \to H_{(2p, \mu)}^{BM}(X_{\mu, v}^\ast(\tau)).
\]

We’re getting closer, because we want a cohomological correspondence on Grassmannians and now we already have a cohomological correspondence supported on affine Deligne-Lusztig varieties!

From Andrea’s talk, we have \( \dim X_{\mu, v}^\ast(\tau^\ast) \leq \langle \rho, \mu \rangle \). Thus \( H_{(\rho, \mu)}^{BM}(-) \) is generated by cycle classes of irreducible components of dimension \( \langle \rho, \mu \rangle \).

Finally, by composing with \( S \), or more precisely with the morphism \( \Xi \) that we used for the precise construction of \( S \) in Ashwin’s talk, we get
\[
\mathcal{C} : \text{Hom}_{\mathcal{O}}(\sigma V_v \otimes V_2 \otimes V_v^\ast, V_\mu) \xrightarrow{\Xi} \text{Corr}_Y(S(\tilde{V}_\tau), S(\tilde{V}_\mu)) \to H_{(2p, \mu)}^{BM}(X_{\mu, v}^\ast(\tau)).
\]

5. Comparison with cohomology classes associated to Satake cycles

There is a second way of getting classes in \( H_{(2p, \mu)}^{BM}(X_{\mu, v}^\ast(\tau)) \) via a result on components of affine Deligne-Lusztig varieties from Dan’s talk, and in this section we relate the two. Recall from Dan’s talk that the components of \( \text{Gr}_1 \) are called Satake cycles, and their set is denoted \( S_{\lambda, \mu, \nu} \). Moreover, there is a natural injection into Mirkovic–Vilonen cycles,
\[
i : S_{(\nu, \mu)|\tau+\sigma v} \to \mathcal{M}_\mu(\tau + \sigma v).
\]
Start with \( a \in S_{(\nu, \mu)|\tau+\sigma v} \) and \( b = i(a) \) satisfying some condition (see Dan’s talk).

There was then a closed subspace \( X_{\mu, v}^a(\tau) \subseteq X_{\mu, v}(\tau) \) associated to the Satake cycle.

**Theorem 5.1.** This \( X_{\mu, v}^a(\tau) \) has a unique irreducible component \( X_{\mu, v}^{b, x_0} \) (see Dan’s talk) of maximal dimension: \( \langle \rho, \mu - v \rangle \).

Claim (Prop. 3.1.10 of XZ): Given such \( a \in S_{(\nu, \mu)|\lambda+\nu} \), we can associate an element \( f_a \in \text{Hom}_{\mathcal{O}}(\sigma V_v \otimes V_2 \otimes V_v^\ast, V_\mu) \) in a natural way.

To \( f_a \) like above, we have thus two different associated classes in \( H_{(\rho, \mu - v)}^{BM}(X_{\mu, v}(\tau)) \):

One given by \( \mathcal{C}(f_a) \), one given by the image of \( [X_{\mu, v}^{b, x_0}] \in H_{(\rho, \mu - v)}^{BM}(X_{\mu, v}(\tau)) \). The main point of this section is that the two agree:

**Proposition 5.2.** Start with a Satake cycle \( a \in S \) satisfying some conditions on \( b = i(a) \). Then
\[
\mathcal{C}(f_a) = [X_{\mu, v}^{b, x_0}].
\]

The proof of this puts us on the right track to prove \( S = T \): there is a way of relating the excursion correspondence \( S(f_a) \) of shtukas associated to \( f_a \) to a sort of excursion correspondence \( \mathcal{C}^{Gr}(f_a) \) supported on (some product of) Grassmannians. Namely, there is a third intermediate correspondence, equipped with natural maps to both \( S(f_a) \) as well as to the excursion correspondence \( \mathcal{C}^{Gr}(f_a) \) on \( \text{Gr}_v \). The precise diagram looks roughly like this:
where the bottom three correspondences are precisely the excursion correspondence $S(f_\alpha)$, the top line are correspondences between Grassmannians which compose to $C^{Gr}(f_\alpha)$, and the line in the middle is some complicated correspondence comparing the two. If you really want more detail, this is on the bottom of page 102 of [XZ].

6. Back to the proof of $S = T$

The last proof gave a way to compare $S(f)$ to correspondences of Grassmannians. Now we can finish discussing the proof of $S = T$. If we specialize the above zig-zag diagram to $\mu = \tau$ and $v = \sigma v$, then it is easy to see that the top row becomes exactly the correspondence $\Gamma_{\sigma \times \text{id}}^\# \circ \delta_\mathcal{F}$ for $\mathcal{F} = \text{Sat}(V_\nu)$, where Sat is the geometric Satake considered as a morphism $\text{Rep}(\hat{G}) \to P_{L^+ G}(\text{Gr})$. More precisely, the correspondence on the top left of the above diagram equals $\delta_\mathcal{F}$ and the composition of second and third correspondence on top equals $\Gamma_{\sigma \times \text{id}}^\#$. This is precisely what we wanted to apply the trace formula to! This gives the desired equation

$$S(\chi_{V_\nu}) = \left( x \mapsto \sum (-1)^i \text{tr}(\phi|\text{Sat}(V_\nu)_x) \right),$$

finishing the proof of $S = T$.

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