AFFINE GRASSMANNIANS

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A talk in the London number theory study group. These are notes taken by Carl Wang-Erickson (c.wang-erickson@imperial.ac.uk) that have been edited by the speaker.

0. Motivation

Let $O$ be a DVR, with residue field $k$ and fraction field $K$. Let $G/O$ be a smooth affine group scheme.

The idea of affine Grassmannians is to put a geometric structure on $G(K)/G(O)$. Via the function-sheaf correspondence, automorphic forms have interpretations as perverse sheaves here. Then you can do geometric Langlands and geometric Satake, which are upcoming topics in this seminar.

1. Equal-characteristic affine Grassmannians

Fix a field $k$, let $O = k[[t]]$, and let $K = k((t))$. Let $G/O$ be a smooth affine group scheme. We work on the site $(\text{Aff}_k)_{\text{fpqc}}$ of affine schemes over $k$ with the fpqc topology and represent objects by $k$-algebras $R$.

**Definition 1.1.** Define the jet group $L^+ G$ and loop group $L G$ to be

$$L^+ G : R \mapsto G(R[[t]])$$

$$L G : R \mapsto G(R((t))).$$

These are sheaves.

**Proposition 1.2.**

(1) $L^+ G$ is representable by an affine scheme.

(2) $L G$ is representable by an ind-affine scheme.

(3) $L^+ G \subset L G$ is a closed subfunctor.

**Example 1.3.** For $G = \mathbb{G}_a$, we have $L^+ G(R) = R[[t]]$ and $L G(R) = R((t))$. Thus $L^+ G$ is representable by $(\mathbb{A}^1_k)^{\geq 0}$ and $L G$ is representable by $\varprojlim_i (\mathbb{A}^1_k)^{\geq -i}$.  

**Definition 1.4.** The affine Grassmannian $\text{Gr}_G$ is the fpqc-quotient

$$\text{Gr}_G := L G / L^+ G.$$

A priori this is only a stack, but it is easy to see there are no automorphisms. There is a moduli interpretation for $\text{Gr}_G$. For $R$ a $k$-algebra, let

$$D_R = \text{Spec } R[[t]], \quad D^*_R = \text{Spec } R((t)).$$
Consider the moduli problem
\[
\mathcal{M}_G : R \mapsto \left\{ (\mathcal{E}, \beta) \mid \mathcal{E} \to D_R \text{ is a } G\text{-torsor}, \beta : \mathcal{E}|_{D_R} \sim \mathcal{E}^0|_{D_R} \right\},
\]
where \(\mathcal{E}^0\) denotes the trivial bundle \(\mathcal{E}^0 := G \times D_R\).
We observe that \(LG = \text{Aut}(\mathcal{E}^0|_{D_R})\) acts on \(\mathcal{M}_G\) by changing \(\beta\). Furthermore, we have this theorem.

**Theorem 1.5.** \(\text{Gr}_G\) is canonically and \(LG\)-equivariantly isomorphic to \(\mathcal{M}_G\).

**Proof.** The map is given by sending the coset of \(A\) in \(LG/L^+G\) via
\[
A \cdot L^+G \mapsto (\mathcal{E}^0, A).
\]
For injectivity, observe that if
\[
(\mathcal{E}^0, A) \phi \to (\mathcal{E}^0, B),
\]
then \(B^{-1}A = \phi \in \text{Aut}(\mathcal{E}^0) = L^+G\). For surjectivity, it suffices to show that every \(G\)-torsor \(\mathcal{E}\) on \(D_R\) is trivial fpqc-locally on \(R\). Suppose that we have a trivialization over the special fiber, i.e. a section of \(\mathcal{E}|_{\text{Spec } R} \to \text{Spec } R\). Since \(\mathcal{E}\) is a \(G\)-torsor on \(D_R\) and \(G\) is smooth, \(\mathcal{E} \to D_R\) is smooth. Thus, any section over the special fiber extends to all of \(D_R\). We have proved that if \(\mathcal{E} \to D_R\) is trivial over the special fiber, then it is trivial. Now in general \(\mathcal{E}\) might not be trivial over the special fiber, but again by smoothness it always becomes trivial over some fpqc cover \(\text{Spec } R' \to \text{Spec } R\). \(\square\)

**Example 1.6.** For \(G = \text{GL}_n\), we have an equivalence between \(G\)-torsors and rank \(n\) vector bundles via
\[
\mathcal{T} \mapsto \mathcal{T} \times_{\text{GL}_n} \mathcal{O}_X^{\oplus n} \cong (\mathcal{T} \times \mathcal{O}_X^{\oplus n})/\text{GL}_n
\]
\[
\text{Isom}(\mathcal{O}_X^{\oplus n}, V) \leftrightarrow V.
\]
So \(\text{Gr}_{\text{GL}_n}\) classifies rank \(n\) vector bundles on \(D_R\) with a trivialization on \(D_R^\circ\).
Consequently, \(\text{Gr}_{\text{GL}_n}(R)\) is the set of finite projective \(R[[t]]\)-modules \(\Lambda \subset R(t)^{\oplus n}\) such that \(\Lambda[1/t] = R(t)^{\oplus n}\).

**Theorem 1.7** ([2], 1.2.2). If \(G/O\) is reductive, then \(\text{Gr}_G\) is representable by a strict ind-projective scheme, that is, \(\text{Gr}_G = \varinjlim_{X \to N} X_N\) with \(X_N\) projective and with transition maps being closed immersions.

We reduce to the case \(G = \text{GL}_n\) using this lemma.

**Lemma 1.8.** If \(G/O\) is reductive, then there exists a closed embedding \(G \hookrightarrow \text{GL}_n\) such that the induced map \(\text{Gr}_G \hookrightarrow \text{Gr}_{\text{GL}_n}\) is a closed immersion.

The idea for \(G = \text{GL}_n\) is to filter the affine Grassmannian by subfunctors bounding the denominator of the lattice. For the moment, write \(G := \text{Gr}_{\text{GL}_n}\). The subfunctors are
\[
\text{Gr}^{(N)}(R) := \{ \Lambda \in \text{Gr}(R) : t^N \Lambda_0 \subset \Lambda \subset t^{-N} \Lambda_0 \},
\]
where \(\Lambda_0 := R[t]/t^{\oplus n}\) and \(N \in \mathbb{Z}_{\geq 1}\).
Now write \(V_N := t^{-N}k[[t]]/t^Nk[[t]],\) which is a finite dimensional \(k\)-vector space.

**Lemma 1.9.** Mapping \(\Lambda \mapsto t^{-N} \Lambda_0/\Lambda\) induces an isomorphism from \(\text{Gr}^{(N)}(R)\) to the set of \(R[[t]]\)-quotient modules of \(V_N \otimes_k R\) which are projective as \(R\)-modules.

Now this set of modules (in the lemma) is representable by the subfunctor of (the conventional Grassmannian) $\text{Gr}(V_N)$ classifying subspaces preserved by $t$.\footnote{By “conventional Grassmannian,” we mean the disjoint union over the integers 0 to $\dim_k V_N$ of the Grassmannian of subspaces of that dimension.}

This subfunctor is a closed subscheme by the following argument. Consider $t \in H^0(\text{Gr}(V_N), \text{End}(V_N))$. There is a map $\text{End}(V_N) \rightarrow \text{Hom}(P, Q)$, where $P$ is the universal subbundle of $V_N$ over $\text{Gr}(V_N)$ and $Q := V_N/P$ is the universal quotient bundle. Then the vanishing of the section of $H^0(\text{Gr}(V_N), \text{Hom}(P, Q))$ induced by $t$ parametrizes subspaces which are preserved by $t$.

Corollary 1.10. $\text{Gr}^{(N)}$ is projective, and
\[
\text{Gr} \cong \varinjlim_N \text{Gr}^{(N)}
\]
is ind-projective.

As this will be important for the mixed characteristic case, we describe the canonical ample line bundle on $\text{Gr}$. On $\text{Gr}^{(N)}$ we have a canonical ample line bundle by pulling back the ample line bundle $\det(Q)$ (for $Q$ as we just defined above) on $\text{Gr}(V_N)$. This line bundle is compatible with the transition maps $\text{Gr}^{(N)} \rightarrow \text{Gr}^{(N+1)}$, so we get a line bundle $\mathcal{L}$ on $\text{Gr}$. This line bundle can be described as follows: For $\Lambda \subset \mathbb{R}(\!(t)\!)^{\otimes n}$ corresponding to an $R$-valued point $f_\Lambda : \text{Spec } R \rightarrow \text{Gr}$ choose an $N$ such that $\Lambda \subset t^{-N} \Lambda_0$, then
\[
f_\Lambda^* \mathcal{L} = \det(t^{-N} \Lambda_0/\Lambda).
\]
If $M \geq N$, then
\[
det(t^{-M} \Lambda_0/\Lambda) = \det(t^{-M} \Lambda_0/t^{-N} \Lambda_0) \otimes \det(t^{-N} \Lambda_0/\Lambda).
\]
As $t^{-M} \Lambda_0/t^{-N} \Lambda_0$ is free, its determinant is a trivial line bundle, so $f_\Lambda^* \mathcal{L}$ is independent of our choice of $N$.

2. Mixed characteristic affine Grassmannian

Let $k$ be a perfect field of characteristic $p$. Let $\mathcal{O} = W(k)$ and $K = W(k)[1/p]$.

Remark 2.1. There is a version for ramified Witt vectors. Since most things are the same, we discuss the unramified version here.

Let $\mathcal{G}/\mathcal{O}$ be a smooth affine group scheme.

The analogue of the equal characteristic case would be
\[
L^+ \mathcal{G} : R \mapsto \mathcal{G}(W(R))
\]
\[
L \mathcal{G} : R \mapsto \mathcal{G}(W(R)[1/p]).
\]

However, taking Witt vectors is pathological for non-perfect $k$-algebras, so we restrict to the site $\text{Perf} \subset (\text{Aff}_k)_{\text{fppf}}$ of perfect affine schemes.

As in the equal characteristic case,

- define $\text{Gr}_\mathcal{G} := L \mathcal{G}/L^+ \mathcal{G}$.  


• set \( D_R = \text{Spec} W(R), \) \( D_R^\vee := \text{Spec} W(R)[1/p], \) then \( \text{Gr}_G \) is isomorphic to the moduli problem

\[
\mathcal{M}_G : R \mapsto \left\{ (\mathcal{E}, \beta) \mid \mathcal{E} \to D_R \text{ is a } G\text{-torsor}, \beta : \mathcal{E}|_{D_R^\vee} \sim \to \mathcal{E}|_{D_R^\vee} \right\},
\]

**Example 2.2.** \( \text{Gr}_{\text{GL}_n} \) classifies finite projective \( W(R)\)-modules \( \Lambda \) in \( W(R)[1/p]^{\oplus n} \) such that \( \Lambda[1/p] = W(R)[1/p]^{\oplus n} \).

• if \( G \) is reductive, there is a closed embedding \( G \to \text{GL}_n \) such that \( \text{Gr}_G \to \text{Gr}_{\text{GL}_n} \) is a closed embedding.

**Theorem 2.3** ([I], 9.6). \( \text{Gr}_G \) is representable by a strict colimit of perfections of projective schemes.

By the result mentioned above, we can reduce to \( G = \text{GL}_n \), writing \( \text{Gr} := \text{Gr}_{\text{GL}_n} \).

We have, for \( N \geq 1 \),

\[
\text{Gr}^{(N)}(R) = \{ \Lambda \in \text{Gr}(R) \mid p^N \Lambda_0 \subset \Lambda \subset p^{-N} \Lambda_0 \},
\]

where \( \Lambda_0 = W(R)^{\oplus n} \).

The main difference relative to the equal characteristic case is that

\[
Q = p^{-N} \Lambda_0 / A \text{ might not be an } \mathbb{R}\text{-module},
\]

so we cannot embed \( \text{Gr}^{(N)} \) directly into a Grassmannian. The idea is to find an ample line bundle on \( \text{Gr}^{(N)} \) nonetheless: Roughly, we consider the \( p \)-adic filtration on \( Q \)

\[
0 \subset p^n Q \subset p^{n-1} Q \subset \cdots \subset pQ \subset Q
\]

and want to define

\[
\mathcal{L} := \bigotimes_{i=0}^{n-1} \det(p^i Q/p^{i+1} Q).
\]

This is not quite the correct definition, because \( p^i Q/p^{i+1} Q \) might not be projective as an \( \mathbb{R}\)-module. In order to get the correct definition we work on the Demazure resolution

\[
\psi : \tilde{\text{Gr}}^{(N)} \to \text{Gr}^{(N)},
\]

i.e. \( \tilde{\text{Gr}}^{(N)} \) classifies filtrations \( F^\bullet Q \) on \( Q \) such that \( F^i Q/F^{i+1} Q \) is a projective \( \mathbb{R}\)-module.

This resolution is what Bhatt–Scholze call a “\( v \)-cover”: it is not fpqc, but looks like a sequence of “blow-ups,” where you allow arbitrary proper geometrically connected spaces as exceptional divisors rather than just \( \mathbb{P}^1 \). Why is it good enough?

Well we have a canonical ample line bundle

\[
\mathcal{L} := \bigotimes \det(F^i Q/F^{i+1} Q)
\]

on \( \tilde{\text{Gr}}^{(N)}, \) which we want to descend.

**Proof sketch for Theorem 2.3.**

1. The \( v \)-topology is not subcanonical on the full category of affine \( k\)-schemes, but it is so on Perf. This implies that vector bundles descend along \( v \)-covers.
2. Show that \( \mathcal{L} \) descends along \( \psi \). This is equivalent to showing that it is trivial on geometric fibers of \( \psi \). This essentially means that \( \bigotimes \det(F^i Q/F^{i+1} Q) \) only depends on \( Q \) and not on the chosen filtration \( F^\bullet Q \).
(3) Show that the descended line bundle is ample.

It is then a formal consequence that $\text{Gr}^{(N)}$ is the perfection of a projective scheme. □

3. SCHUBERT VARIETIES

We treat the equal and mixed cases simultaneously in this discussion.

Let $k$ a field, as we specified in each of the two cases above. Likewise, set $\mathcal{O}$ and $K$. Assume that $G/O$ is split, and choose $T \subset B \subset G$ a maximal torus and Borel.

If $k'/k$ is a (perfect) field extension, write $\mathcal{O}' = k'[t]$ (or $W(k')$), and $K' = \text{Frac} \mathcal{O}'$. The Cartan decomposition gives a canonical isomorphism

$$G(O)/G(K)/G(O') \cong X^*(T)/W = X_*(T)^+,$$

where $W$ is the Weyl group, which is in turn isomorphic to $G(O')/G(K')/G(O')$.

For $(E, \beta) \in \text{Gr}_G(k')$, fix a trivialization $\phi : E \sim E^0$. Then

$$\phi \beta^{-1} \in \text{Aut}(E^0|_{D^+_x}) = G(K'),$$

and we define the type or relative position $\text{inv}(\beta)$ of $\beta$ as the image of $\phi \beta^{-1}$ in $X_*(T)^+$. For different choices $\phi$ of trivialization, the elements $\phi \beta^{-1}$ form a single $G(O')$-coset in $G(K')$, so $\text{inv}(\beta)$ does not depend on such a choice.

For $(E, \beta) \in \text{Gr}_G(R)$, $x \in \text{Spec} R$, let

$$\text{inv}_x(\beta) := \text{inv}(\beta|_x).$$

For some $\mu \in X_*(T)^+$, we say that $E$ is of type $\leq \mu$ (resp. $= \mu$) if $\text{inv}_x(\beta) \leq \mu$ (resp. $\text{inv}_x(\beta) = \mu$) for all $x \in \text{Spec} R$, according to the Bruhat ordering.

The following lemma gets the theory of Schubert varieties off the ground.

Lemma 3.1. For $\mu \in X_*(T)^+$ and $(E, \beta) \in \text{Gr}_G(R)$, then

$$(\text{Spec} R)_{\leq \mu} = \{ x \in \text{Spec} R | \text{inv}_x(\beta) \leq \mu \} \subset \text{Spec} R$$

is closed.

Definition 3.2. The Schubert variety $\text{Gr}_{\leq \mu}$ is defined as the closed subset

$$\{(E, \beta) \in \text{Gr}_G \mid \text{inv}(\beta) \leq \mu \} \subset \text{Gr}_G$$

with reduced induced subscheme structure.

By Lemma 3.1, $\text{Gr}_{\leq \mu}$ is closed in $\text{Gr}_G$, and hence (perfectly) projective. Note that we need not say "ind-(perfectly) projective" because it is contained in one of the $\text{Gr}^{(N)}$ for some embedding $G \hookrightarrow \text{GL}_n$ and $N \gg 0$.

Definition 3.3. The Schubert cell $\text{Gr}_\mu$ is defined as

$$\{(E, \beta) \in \text{Gr}_G \mid \text{inv}(\beta) = \mu \} \subset \text{Gr}_G.$$

²Here the statements in (...) apply to the mixed characteristic case in parentheses.
We observe that
\[ \text{Gr}_\mu = \left( \text{Gr}_{\leq \mu} \setminus \bigcup_{\lambda < \mu} \text{Gr}_{\leq \lambda} \right) \subset \text{Gr}_{\leq \mu} \]
is open. It is also (Zariski) dense by [2, 2.1.5 (2)] in equal characteristic and [3, 1.23 (3)] in mixed characteristic.

This is the main theorem about Schubert cells/varieties.

**Theorem 3.4** ([2, 2.1.5(1)] in equal char.; [3, 1.23] in mixed char.). \( \text{Gr}_\mu \) is a single \( L^+G \)-orbit on \( \text{Gr}_G \), hence \( \text{Gr}_\mu \) is (perfectly) smooth. In particular, if \( \mu \) is minuscule, then \( \text{Gr}_\mu = \text{Gr}_{\leq \mu} \) is (perfectly) smooth and projective.

**Proof sketch.** For \( \mu : G_m \to T \), we get \( \mu(t) \in L^G(k) \). Let \( x = \mu(t) \cdot L^G \) be considered as a point of \( \text{Gr}_G \). The stabilizer of \( x \) in \( L^+G \) is
\[ L^+G \cap \mu(t) L^G \mu(t)^{-1}. \]

Therefore the induced map
\[ L^+G/\text{Stab}_x \longrightarrow \text{Gr}_G, \quad g \mapsto gx \]
is a locally closed embedding.

We also observe that by the Cartan decomposition, a geometric point \((E, \beta)\) is in the image of this map if and only if its type is equal to \( \mu \). Then use the fact that the reduced induced structure on a homogeneous space is smooth. \( \square \)

For minuscule cocharacters, the mixed and equal characteristic cases are closely comparable.

**References**


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