Uniqueness of Polish Group Topologies

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Outline

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Definition of a Polish Group

Polish Group is a topological group that is separable and metrizable by a complete metric.

Polish Groups are Ubiquitous

Many naturally occurring topological groups are Polish.

Polish Groups are Ubiquitous

In Analysis:

- Separable Banach spaces
- Unitary group of a separable Hilbert space
- Compact Lie groups

Polish Groups are Ubiquitous

In Algebra:

- Permutation groups
- Automorphism groups of first order structures
- Galois groups

Polish Groups are Ubiquitous

In Geometry and Topology:

- Isometry group of a complete separable metric space
- Autohomeomorphism groups of manifolds
  - Autohomeomorphism group $Homeo(S^1)$ of the unit circle, with compact-open topology

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General Question

Algebraic vs Topological Structure

In a topological group,

- How do the topology and algebra interact?
- More precisely, how strongly does the algebraic structure constrain the topology?

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Uniqueness of Topology and Automatic Continuity

- (U) When does a group admit a unique Polish group topology?
- (AC) When is a homomorphism between two topological groups necessarily continuous?

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- Banach aglebras: *Automatic Continuity in Banach Algebras, Dales [1]*
- Compact Lie Groups: van der Waerden's Continuity Theorem
- Profinite Groups: Serre's conjecture

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The Key: Definability of Sets

The key is to understand which sets are ‘definable’, or ‘computable’, both algebraically and topologically (Borel).

A Classical Resut of Mackey

- Guarantees uniqueness of a Polish group topology.
- Requires algebraically definable sets that are *always* Borel.
- Problem: Most algebraically definable sets are always analytic, but not obviously Borel.

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Are Algebraic Sets Necessarily Borel?

Answer: No!

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Improved Uniqueness Result

Theorem 2 *(PG,BP) If G is a Polish group with a neighborhood base at 1 of sets that are always analytic, then G has a unique Polish group topology.*

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Applications: Compact Lie Groups

Theorem 3 *(BP,PG) If G is a compact, connected simple Lie group with trivial center (e.g. $SO(3,\mathbb{R})$), then G has a unique Polish group topology.*

Applications: Profinite Groups

Theorem 4 *(PG,BP) If G is a finitely generated profinite group, then G has a unique Polish group topology.*

References

H. G. Dales, *Banach algebras and automatic continuity*, Oxford University Press, New York, 2001.

P. Gartside and B. Pejić, *The complexity of the set of squares in the homeomorphism group of the unit circle*, Submitted for publication. http://www.math.pitt.edu/~boka/research.html (2006).