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# 1 Background and Motivation

## 1.1 Objects: Polish Groups

Topological group is a group  $(G, \cdot)$  together with a topology on  $G$  such that the group operations  $(x, y) \mapsto xy$  and  $x \mapsto x^{-1}$  are continuous. My research is concerned with a special kind of topological groups, known as the Polish groups.

### Definition of a Polish Group

*Polish Group* is a topological group that is separable and metrizable by a complete metric.

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Bojana Pejić, Dr. Paul Gartside

Uniqueness of Polish Group Topologies

## Polish Groups are Ubiquitous

Many naturally occurring topological groups are Polish. To understand a mathematical object one often needs to understand its symmetries, and groups of symmetries of reasonably small objects come naturally equipped with a topological structure making them into a Polish group.

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Uniqueness of Polish Group Topologies









## Uniqueness of Topology and Automatic Continuity

- (U) When does a group admit a unique Polish group topology?
- (AC) When is a homomorphism between two topological groups necessarily continuous?

In other words, when is a topology of certain type (Polish, locally compact separable metric) determined by the group structure?

## Uniqueness of Topology and Automatic Continuity

- ▶ (U) When does a group admit a unique Polish group topology?
- ▶ (AC) When is a homomorphism between two topological groups necessarily continuous?





## A Classical Result of Mackey

- Guarantees uniqueness of a Polish group topology.
- Requires algebraically definable sets that are *always* Borel.
- Problem: Most algebraically definable sets are always analytic, but not obviously Borel.

The difficulty in applying Mackey's result is deciding which sets are Borel in *any* Polish group topology. Algebraically defined sets are good candidates (as their definition does not involve any particular topology). It is, therefore, of interest to know which algebraically defined sets in  $G$  must be Borel in any Polish group topology on  $G$ .

## A Classical Result of Mackey

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## 2 Our Results

### 2.1 Main Results

#### Are Algebraic Sets Necessarily Borel?

Answer: No!

**Theorem 1** (*BP, PG [2]*) *The set of squares in  $\text{Homeo}(S^1)$  is completely analytic, and hence not Borel.*

Though the statement of this result is simple, the proof is quite difficult and involves techniques from descriptive set theory. This is my most significant result.

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Uniqueness of Polish Group Topologies

## Improved Uniqueness Result

**Theorem 2 (PG,BP)** *If  $G$  is a Polish group with a neighborhood base at 1 of sets that are always analytic, then  $G$  has a unique Polish group topology.*

- Guarantees uniqueness of the Polish group topology.
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- This result avoids the problem of algebraic sets not being necessarily Borel.

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