The Circular Squares are Wild

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Summer Topology Conference at Denison, July 2005

General Problem in Polish Groups

Polish groups are separable completely metrizable topological groups.

- Many naturally occurring topological groups are Polish: autohomeomorphism groups, isometry groups, Lie groups, automorphism groups of first order structures...

Fundamental Problem (Automatic Continuity).
Given an algebraic homomorphism between two Polish groups, when can we conclude that the homomorphism must be continuous?

- The key is to understand which sets are 'definable' both algebraically and topologically.

- In this context, 'definable topologically' means Borel.

Examples of sets defined algebraically

- Conjugacy Classes

$$[a] = \{bab^{-1} \mid b \in G\}$$

- Commutators

$$\{aba^{-1}b^{-1} \mid a, b \in G\}$$

- Squares

$$G^2 = \{a^2 \mid a \in G\}$$

Fact. Conjugacy classes are always Borel.

Question. What about the squares, commutators?

We looked into the squares and...

... we found that:

- in S_∞ the squares are Borel
- in $Hom(I)$ the squares are Borel
- in $Hom(S^1)$ the squares are **not** Borel

Here:

$$S_\infty = \{\text{permutations of } \mathbb{N}\} \subset \mathbb{N}^\mathbb{N}$$

$$Hom(I) = \{\text{homeomorphisms of } I\} \subset C(I, I)$$

$$Hom(S^1) = \{\text{homeomorphisms of } S^1\} \subset C(I, I)$$

In this talk I will present the result that the squares in the $Hom(S^1)$ are not Borel.

Borel and Analytic Sets

Borel sets are the σ-algebra of open sets.

- Open, closed, G_δ, F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta}$,... sets are all examples of Borel sets.
- Borel sets form a strictly increasing hierarchy:

$$\text{Borel sets} = \bigcup_{\alpha<\omega} \Sigma_\alpha$$

Analytic sets are the continuous images of Borel sets.

- Not all analytic sets are Borel.

Completely Analytic Sets

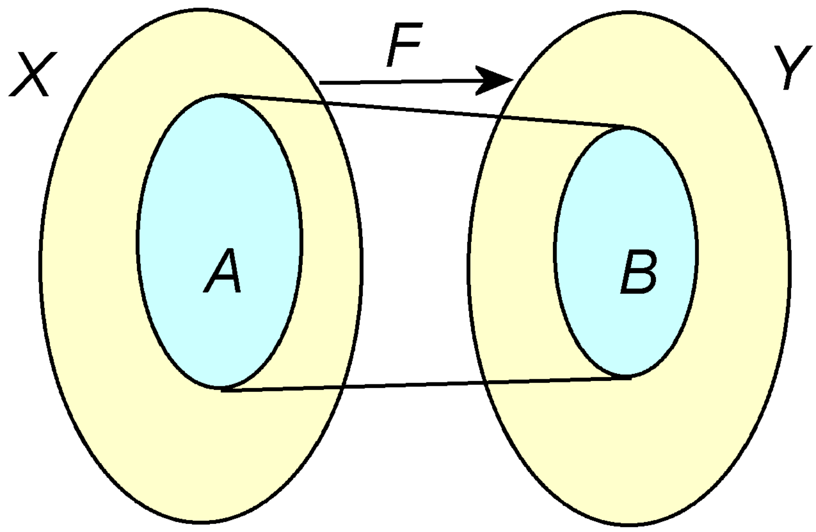
Completely Analytic Analytic set $B \subset Y$ is complete if whenever $A \subset X$ is analytic, there is a continuous function

$$F : X \to Y$$

such that

$$F^{-1}(B) = A.$$

- F is called a *continuous reduction* of A to B, and we say that A can be *reduced* to B.



To show our set is not Borel, we use:

Theorem. B completely analytic $\Rightarrow B$ not Borel

Proof. Suppose B is Borel.

Then $B \in \Sigma_\beta$, some $\beta < \omega$.

Pick a Borel set $A \in \Sigma_\alpha \setminus \Sigma_\beta$, where $\alpha > \beta$.

$\exists$ continuous $F : X \to Y$ such that $F^{-1}(B) = A$.

But Σ_β is closed under continuous preimages.

Thus, $A \in \Sigma_\beta$ – a contradiction!

Squares in $Hom(S^1)$ are wild

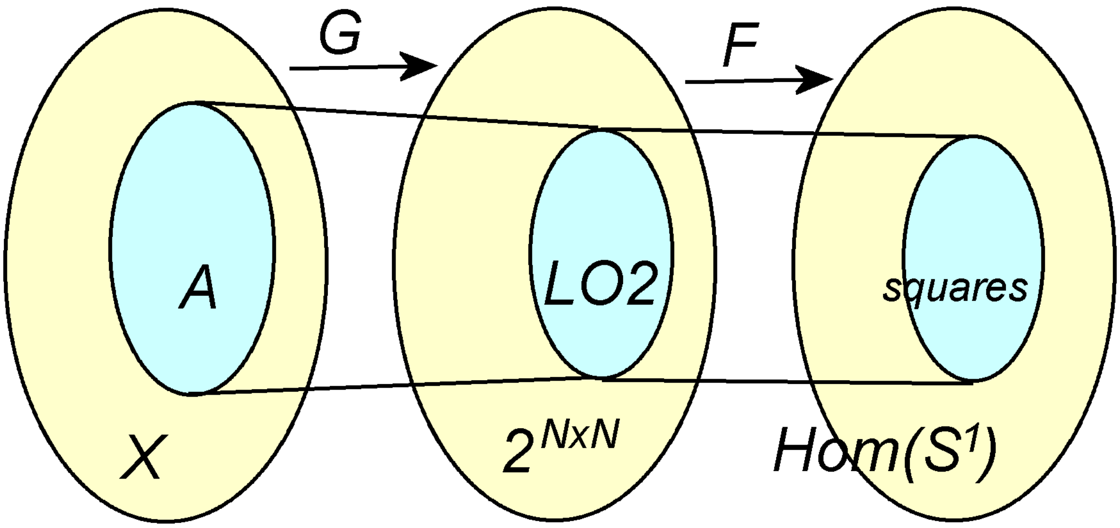
Theorem. The set of squares

$$Hom(S^1)^2 = \{f^2 \mid f \in Hom(S^1)\}$$

is completely analytic (and thus not Borel).

Proof. Clearly analytic, since $f \mapsto f^2$ continuous.

To show completeness, a standard technique is to reduce an already known complete analytic set to the given set.



LO and $LO2$

Let $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$ code the relation $<_\alpha$ on $\mathbb{N}$ the following way:

$$\alpha(n,m) = 1 \Leftrightarrow n <_\alpha m.$$

$$\begin{aligned} LO &= \{\alpha \in 2^{\mathbb{N}\times\mathbb{N}} \mid \alpha \text{ codes a linear order}\} \\ LO2 &= \{\alpha \in LO \mid \alpha \text{ codes order of the form } I + I\} \end{aligned}$$

Facts.

- LO is Borel
- $LO2$ is completely analytic (Beleznay, 1998)

Characterization of Squares

Let

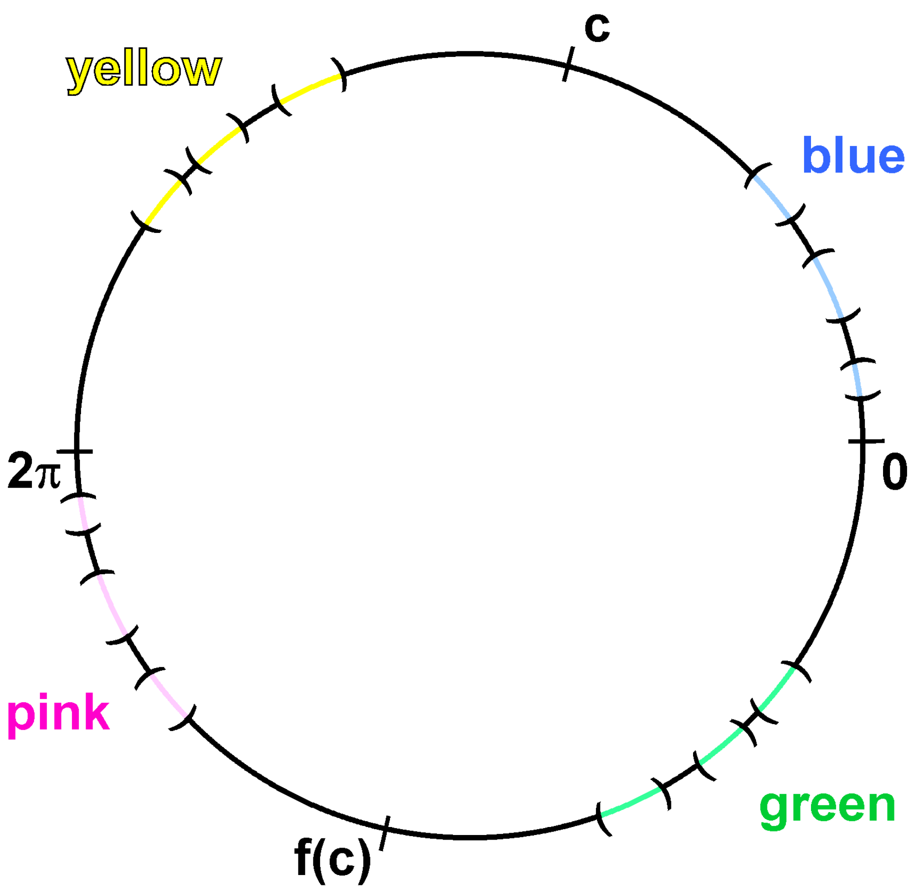
$$\mathcal{M} = \{f \in Hom(S^1) \mid f(0) = \pi, f(\pi) = 0,$$
$$f \text{ rotates each point by at least } \pi\}$$

Characterization Lemma. For a homeomorphism $f \in \mathcal{M}$ the following are equivalent:

(a) f is a square,

(b) The intervals in $(0, \pi)$ which are not fixed by f^2 form an order of the type $I + I$.

Pretty Picture



- f rotates the black points by π, takes blue to pink, yellow to green, pink to blue and green to yellow.

- A square root of f takes blue to yellow, yellow to pink, pink to green and green to blue.

Continuous Reduction

Theorem. There is a continuous function

$$F : LO \to \mathcal{M} \subset Hom(S^1)$$

such that

$$F(\alpha) \text{ is a square} \Leftrightarrow \alpha \in LO2.$$

Proof. Uses the Characterization Lemma.

In other words, F is a continuous reduction of $LO2$ to $Hom(S^1)^2$.