Math 2370 Matrices and Linear Operators
Solutions and Hints to Practice Problems 6
Bojana Pejić

Problem 1. There are n choices for $p(1)$, $n-1$ choices for $p(2)$, ... , 1 choice for $f(n)$, which gives $n(n-1)\cdots 1 = n!$ different permutations. If $(a_1\ a_2\ \ldots\ a_k)$ is a k-cycle, there are n choices for a_1, $n-1$ choices for a_2, ... , $n-k+1$ choices for a_k; this gives $n(n-1)\cdots(n-k+1)$ choices for a k-cycle. However, not all of these cycles are distinct, for example $(1\ 2\ \ldots\ k)$ is the same cycle as $(2\ 3\ \ldots\ k\ 1)$, or $(3\ 4\ \ldots\ k-1\ 1\ 2)$ etc. Thus each cycle was counted k times, so the total number of distinct k- cycles is $\frac{n(n-1)\cdots(n-k+1)}{k}$. A transposition is a 2-cycle, so there are $\frac{n(n-1)}{2}$ distinct transpositions.

Problem 2. We know that each permutation can be written as a product of (distinct) cycles. Thus, it is sufficient to show that each cycle can be written as a composition of transpositions of type $(1\ k)$. Note that $(1\ a_1)(1\ a_k)(1\ a_{k-1})\cdots(1\ a_2)(1\ a_1) = (a_1\ a_2\ a_3\ \cdots\ a_{k-1}\ a_k)$, and we are done.

Problem 3. If the j-th component of each of the vectors $a_1, a_2, \ldots, a_n$ is 0, then they all belong to the vector subspace

$$\{(x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_n) | x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n \in \mathbb{K}\}.$$

But this subspace has dimension $n-1$ (there is an obvious basis of $n-1$ vectors), so our n vectors must be linearly dependent. It follows from the properties of the determinant that $D(a_1, a_2, \cdots, a_n) = 0$.

Problem 4. Remark. In the recitation we gave a motivation for the solution that follows. Recall, first we realized that we need to find two vectors e_1 and e_2 such that $Se_1 = e_1$ and $Se_2 = 0$, because then the matrix of S with respect to the basis $\{e_1, e_2\}$ would have the desired form : $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We then drew some pictures and noticed that for a suitable vector w, vectors Sw and $w - Sw$ are good candidates for e_1 and e_2. With this observation, our problem is practically solved, but we still have to write up our solution carefully and justify the 'little' details, like the fact that $\{e_1, e_2\}$ indeed *is* a basis.

The linear map S is a projection, so $S^2 = S$. Suppose $S \neq 0$ and $S \neq I$. Fix a vector $w \neq 0$ in $\mathbb{R}^2$ such that $Sw \neq 0$ and $Sw \neq w$. (Why does such a vector exist? Suppose for a contradiction, that for all vectors $w \neq 0$, either $Sw = 0$ or $Sw = w$. Let $w_1 \neq 0$ be such that $Sw_1 = 0$ and let $w_2 \neq 0$ be such that $Sw_2 = w_2$. Then $S(w_1 + w_2) = Sw_1 + Sw_2 = w_2$. So, for $w_3 = w_1 + w_2$, we have neither $Sw_3 = 0$ nor $Sw_3 = w_3$ — a contradiction!)

Let $e_1 = Sw$ and $e_2 = w - Sw$. Claim: $\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$. It is sufficient to show that e_1, e_2 are linearly independent. Suppose $\alpha e_1 + \beta e_2 = 0$. Applying S to both sides of this equation, we get $\alpha Se_1 + \beta Se_2 = 0$, i.e. $\alpha S(Sw) + \beta S(w - Sw) = 0$, i.e. $\alpha Sw + \beta(Sw - Sw) = 0$, so $\alpha Sw = 0$. Since $Sw \neq 0$, get $\alpha = 0$. Then $\beta(w - Sw) = 0$, but since $Sw \neq w$, get $\beta = 0$. This proves that $\{e_1, e_2\}$ is a basis.

Now, $Se_1 = S(Sw) = S^2 w = Sw = e_1$ and $Se_2 = S(w - Sw) = Sw - S^2 w = Sw - Sw = 0$, so S has the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the basis $\{e_1, e_2\}$.

Problem 5. We define a linear map $T : X \to Y$ by

$$T(x) = T(k_1 x_1 + \cdots + k_n x_n) := k_1 y_1 + \cdots + k_n y_n.$$

Then clearly, $Tx_j = 0x_1 + \cdots + 1x_j + \cdots + 0x_n = 0y_1 + \cdots + 1y_j + \cdots + 0y_n = y_j$. This shows that the existence of a linear map T such that $Tx_j = y_j$ for all j. Next, we need to show the uniqueness. Let $S : X \to Y$ be another linear map with the property that $Sx_j = y_j$ for each j. We want to show that then, in fact, $S = T$. Let $x = k_1 x_1 + \cdots + k_n x_n$. Then $S(x) = S(k_1 x_1 + \cdots + k_n x_n) = k_1 Sx_1 + \cdots + k_n Sx_n = k_1 y_1 + \cdots + k_n y_n = T(x)$. Thus, indeed, $S = T$.

Problem 6. Recall that two matrices A and B are similar if and only if there exists an invertible matrix U such that $U^{-1}AU = B$, i.e. $AU = UB$. Writing $U = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix}$, the matrix equation $AU = UB$ with $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{pmatrix}$ yields 16 equations with

20 unknowns:

$$\begin{array}{cccc} x_1 + x_5 = x_2 & x_2 + x_6 = x_3 & x_3 + x_7 = x_4 & x_4 + x_8 = ax_1 + bx_2 + cx_3 + dx_4 \\ x_5 = x_6 & x_6 = x_7 & x_7 = x_8 & x_8 = ax_5 + bx_6 + cx_7 + dx_8 \\ x_{13} = x_{10} & x_{14} = x_{11} & x_{15} = x_{12} & x_{16} = ax_9 + bx_{10} + cx_{11} + dx_{12} \\ -x_9 = x_{14} & -x_{10} = x_{15} & -x_{11} = x_{16} & -x_{12} = ax_{13} + bx_{14} + cx_{15} + dx_{16} \end{array}$$

Using the 12 leftmost equations (that do not involve a, b, c, d) we find that

$$U = \begin{pmatrix} x_1 & x_1 + x_5 & x_1 + 2x_5 & x_1 + 3x_5 \\ x_5 & x_5 & x_5 & x_5 \\ x_9 & x_{10} & -x_9 & -x_{10} \\ x_{10} & -x_9 & -x_{10} & x_9 \end{pmatrix}.$$

All matrices U of the above form satisfy $AU = UB$. We only need *one* such *invertible* matrix to retrieve the values for a, b, c, d. Taking $x_1 = x_5 = x_9 = x_{10} = 1$, it is easily checked that the matrix

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

is invertible (by row reduction, for example). Now $AU = UB$ gives 4 equations in 4 unknowns:

$$\begin{array}{ccccccccc} a & + & 2b & + & 3c & + & 4d & = & 5 \\ a & + & b & + & c & + & d & = & 1 \\ a & + & b & - & c & - & d & = & -1 \\ a & - & b & - & c & + & d & = & 1 \end{array}$$

Solving this system we get $a = -\frac{1}{2}$,$b = \frac{1}{2}$,$c = -\frac{1}{2}$ and $d = \frac{3}{2}$.

Problem 7. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Problem 8. Left to you! To get started, write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$N_T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\},$$

etc. Similarly for R_T.