Math 2370 Matrices and Linear Operators
Solutions and Hints to Practice Problems 13
Bojana Pejić

Here are the solutions to the first 5 questions, we will cover Problems 6-10 in recitation on Thursday. I may not have time to type up the solutions to Problems 6-10 in time before your final exam, so please come to the recitation and the office hour if you have any questions.

Problem 1. Recall that all Hermitian matrices, and more generally all normal matrices, are unitarily diagonalizable. Check that $\begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$ is Hermitian, and $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is normal, so both of these are unitarily similar to a diagonal matrix. The third matrix, $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has minimal polynomial $m(s) = s^2$. Since the minimal polynomial is not a product of distinct linear factors, the matrix is not diagonalizable at all (let alone unitarily). The fourth matrix, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 1 and 2, with corresponding eigenvectors $(1\,0)^T$ and $(1\,1)^T$, respectively. Since these two vectors are not orthogonal, we conclude that the matrix cannot have an orthonormal basis of eigenvectors, i.e. it is not unitarily diagonalizable.

Problem 2. **(a)** T does not map P_n into itself, so it cannot be a self-adjoint operator on P_n.
(b) If D were self-adjoint, we would have $\phi(D(f), g) = \phi(f, D(g))$ for every f and g in P_n. Check that for $f(t) = 1$, and $g(t) = t$, we get $\phi(D(1), t) \neq \phi(1, D(t))$. It follows that D is not self-adjoint.

Problem 3. Recall, in recitation we showed that the rotation of $\mathbb{R}^2$ about the origin through angle α is given by the following matrix with respect to the standard orthonormal basis:

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

We also showed that any orthogonal 2×2 matrix with determinant equal to

1, has the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

for some $\alpha \in [0, 2\pi]$, i.e. any orthogonal 2×2 matrix with determinant 1 represents a rotation with respect to the standard basis.

Suppose O is a 3×3 orthogonal matrix with determinant 1. (We also denote by O the linear map from $\mathbb{R}^3$ into itself, that takes x to $O \cdot x$.) We first show that O has an eigenvalue equal to 1. Let λ be an eigenvalue, and u a corresponding unit eigenvector. Then $|\lambda|^2 = |\lambda|^2(u, u) = (\lambda u, \lambda u) = (Ou, Ou) = (u, O^*Ou) = (u, Iu) = 1$, so $|\lambda| = 1$. The eigenvalues of O are the roots of its characteristic polynomial. Recall that the roots of a polynomial with real coefficients are either real or come in complex conjugate pairs. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of O. If they are all real, then each of them is either 1 or -1 and $\lambda_1\lambda_2\lambda_3 = \det O = 1$. It follows that one eigenvalue must be 1. If the eigenvalues are not all real, we may assume that λ_1 is real, and $\lambda_2 = \overline{\lambda_3}$. Then $\lambda_1 = \lambda_1|\lambda_2|^2 = \lambda_1\lambda_2\overline{\lambda_2} = \lambda_1\lambda_2\lambda_3 = \det O = 1$.

Let e_3 be an eigenvector of unit length associated with the eigenvalue 1, and let e_1, e_2, e_3 be the positively oriented orthonormal basis of $\mathbb{R}^3$. Then the matrix of O with respect this basis is easily seen to be of the form

$$\begin{pmatrix} o_{11} & o_{12} & 0 \\ o_{21} & o_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $Q = \begin{pmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{pmatrix}$ is a 2×2 orthogonal matrix with determinant 1. Thus Q is of the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

for some $\alpha \in [0, 2\pi]$, i.e. Q represents a rotation of the plane $\mathrm{Span}(e_1, e_2)$ about the origin. The matrix O in the basis e_1, e_2, e_3 is thus of the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for some $\alpha \in [0, 2\pi]$. In other words, the matrix O is unitarily similar to this matrix, via the unitary matrix U with columns e_1, e_2, e_3.

If you wanted to find the axis and the angle of a rotation that is represented by a 3×3 orthogonal matrix with determinant 1, you would proceed as follows. Fist find e_3 — a unit eigenvector corresponding to the eigenvalue 1 (solve $Oe_3 = e_3, ||e_3|| = 1$ for e_3). Then $\mathrm{Span}(e_1)$ is the axis of the rotation. Next, find unit vectors e_1 and e_2 such that e_1, e_2, e_3 is a positively oriented orthonormal basis (note: the choice of e_1 and e_2 is not unique). Let $U = [e_1\ e_2\ e_3]$. Then $U^T OU$ is of the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate $U^T OU$ and solve for α to find the angle of rotation.

Problem 4. Find the eigenvalues λ_1, λ_2 (these turn out to be distinct) and corresponding **unit** eigenvectors v_1, v_2 of A. Let Q be the matrix $[v_1\ v_2]$ with columns v_1 and v_2. Then, from the theory we know that $Q^{-1}AQ = D$, where D is the diagonal matrix with entries λ_1, λ_2. Note that since A is self-adjoint, the eigenvectors corresponding to distinct eigenvalues are orthogonal. Thus Q is orthogonal, so $Q^{-1} = Q^T$. So, $Q^T AQ = D$. Thus we have found the required matrix Q. (Remark, if, in a different example, the eigenvalues of A are not distinct, make sure you choose the corresponding eigenvectors v_1, v_2 to be orthogonal).
(I will post what I got for $\lambda_1, \lambda_2, v_1, v_2, U$ soon; my solution is at school, and I am at home now).

Problem 5. This is an immediate consequence of the polarization identity (see Problem 7 in Practice Problems 12). What polarization identity tells us is that the inner product is uniquely determined by its induced norm. If $(v, v)_1 = (v, v)_2$, for all $v \in V$, then the norms induced by the two inner products are equal. The polarization identity then tells us that they must be the same inner product: $(v, w)_1 = (v, w)_2$ for all $v, w \in V$.

If you don't remember the polarization identity, it is just as quick to do this problem directly:

$$(v + w, v + w)_j = (v, v)_j + 2\mathrm{Re}(v, w)_j + (w, w)_i$$

implies that $\mathrm{Re}(v, w)_1 = \mathrm{Re}(v, w)_2$. Similarly,

$$(v + iw, v + iw)_j = (v, v)_j + 2\mathrm{Im}(v, w)_j + (w, w)_j$$

implies that $\mathrm{Im}(v, w)_1 = \mathrm{Im}(v, w)_2$. Thus, $(v, w)_1 = (v, w)_2$. (This is essentially the derivation of the polarization identity.)