Math 2370 Matrices and Linear Operators
Solutions and Hints to Practice Problems 10
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Here are some texts that cover Jordan canonical form: 1. Matrix Analysis, by Horn and Johnson, 2. Linear Algebra, by Hoffman and Kunze, 3. Linear Algebra Done Right (Chapter 8), by Axler and 4. Linear Algebra and Its Applications, Strang. The last two are undergraduate texts.

Problem 1. By the Jordan canonical form theorem, we know that the following two matrices have characteristic polynomial $(s-2)^4$ and minimal polynomial $(s-2)^2$:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(The characteristic polynomial tells us that the matrices must have the eigenvalue 2 with multiplicity 4 — so both matrices must have 2's along the diagonal. The degree of the factor $(s-2)$ in the minimal polynomial (here, 2) tells us the size of the largest Jordan block corresponding to the eigenvalue 2. Thus the possible sizes of the Jordan blocks are $2, 2$ or $2, 1, 1$.)
The matrices are not similar because their Jordan canonical forms are not equal (their Jordan canonical forms are the matrices themselves).

Problem 2. **(a)** Trivially, $\{0\}$ and $\mathbb{R}^2$ are invariant subspaces of T. We look for 1-dimensional linear subspaces invariant under T. Suppose $(x_1\ x_2)^T \neq 0$ is a vector in a 1-dimensional invariant subspace. Then:

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1-\lambda & -1 \\ 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

Since $\det \begin{pmatrix} 1-\lambda & -1 \\ 2 & 2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 4$ is non-zero for all real values of λ, the system 1 has non non-trivial solutions for x_1, x_2. It follows that there are no 1-dimensional subspaces invariant under T.

(b) Start as in (a), but notice that $\lambda^2 - 3\lambda + 4$ does have roots over $\mathbb{C}$. These roots will be the eigenvalues, and the corresponding two eigenspaces are 1-dimensional invariant subspaces.

Problem 3. Each (monic) polynomial over $\mathbb{C}$ factorizes into linear factors: $(s-\lambda_1)^{n_1}\cdots(s-\lambda_k)^{n_k}$. A diagonal matrix with k_i-many entries λ_i on the diagonal has the desired characteristic polynomial.

But what if we only know the coefficients of a monic polynomial, and not its roots? Can we find a matrix such that its characteristic polynomial is exactly the given monic polynomial? The answer is yes.

The following matrix is called the *companion matrix* of the polynomial Given a monic polynomial $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ the companion matrix of $p(s)$, denoted $\mathcal{C}_{p(s)}$, is the $n \times n$ matrix with 1's down the first subdiagonal and minus the coefficients of $p(s)$ down the last column:

$$\mathcal{C}_{p(s)} = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & -a_0 \\ 1 & 0 & \dots & \dots & \dots & -a_1 \\ 0 & 1 & \dots & \dots & \dots & -a_2 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & 1 & -a_{n-1} \end{pmatrix}.$$

Can you show that the characteristic polynomial of this matrix is precisely $p(s)$?

Problem 4. **(a)** (The solution here is different from the one we gave in recitation.) The characteristic polynomial of A is s^3, and since $A \neq 0$ and $A^2 \neq 0$, the minimal polynomial of A is also s^3. It follows that the Jordan canonical form of A must be

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is B. So, A and B are similar.

(b) Similar to (a) — left to you. If you have questions (as with any problem), feel free to ask me during the recitation or office hour (or e-mail me).

Problem 5. Let $x_1, \ldots, x_n$ be a basis for V that consists of generalized eigenvectors of T (such a basis always exists). We will show that each x_i must, in fact, be a true eigenvector. Let $n_i \geq 1$ be the smallest natural number such that $(T - \lambda_i I)_i^n x_i = 0$. Suppose $n_i \geq 2$. Let $w = (T - \lambda_i I)^{n_i - 1} x_i \neq 0$. Show that then $w \in N_{T-\lambda_i I} \cap R_{T-\lambda_i I}$, which is a contradiction. It follows that $n_i = 1$, and so x_i is a true eigenvector.

Problem 6. Fix the canonical basis and write down the matrix of D with respect to it. Use this to find the characteristic polynomial (s^5), eigenvalues (all 0), eigenvectors (constant polynomials). It is clear that $D, D^2, D^3, D^4 \neq 0$, so the minimal polynomial of D is also s^5.

Problem 7. $T^k = 0$ tells us that the polynomial s^k annihilates T. It follows that the minimal polynomial $m(s)$ of T divides s^k, so $m(s) = s^l$, for some $1 \leq l \leq n$. But then $m(s)$ divides s^n, so s^n annihilates T, i.e. $T^n = 0$.