

Math 2370 Matrices and Linear Operators
Solutions and Hints to Practice Problems 1
Bojana Pejić

Remark. This first solution set is unusually long. I often went into great detail, explaining along the way the reasoning behind important steps in the proofs. As the course progresses, I expect the solutions posted will be less detailed, sometimes just hints. Also, I might skip the solutions of the problems well covered in the recitation.

Problem 1. We note that $1 \cdot (2, 3) = (2, 0) \neq (2, 3)$. Thus V (with the given addition and scalar multiplication) fails to satisfy the vector space axiom $1 \cdot x = x$ and is not a vector space.

Problem 2. Necessary and sufficient conditions for $\emptyset \neq V \subset P$ to be a vector subspace of P are

$$\begin{aligned} \forall x, y \in P \quad x + y \in P & \quad (V \text{ is closed under addition}), \\ \forall x \in P \forall k \in \mathbb{R} \quad kx \in P & \quad (V \text{ is closed under scalar multiplication}). \end{aligned}$$

Remark. When you want to prove that V is a vector subspace, you have to show that for *arbitrary* $x, y \in V$ and $k \in \mathbb{R}$ both of the above conditions hold. However, if you want to prove that V is *not* a vector subspace, then you need to give a *specific* example that shows that one of the above conditions fails.

(a) V is not a vector subspace because it is not closed under scalar multiplication.

Remark. You might feel that justifying this fact in general terms (“... if $x(t)$ is positive then $-x(t)$ is negative...”) would be the more proper way to prove this. However, the clearer, easier and actually more proper way is to exhibit a very concrete counterexample: a specific polynomial $x \in V$ and a specific scalar k for which $kx \notin V$. Of course, these are not too hard to think of, x could be almost anything and k should be negative. So, this is what we write:

With $x(t) = t$ ($x \in V$) and $k = -1$, $kx(t) = -t$, so clearly, $kx \notin V$.

(b) V is a vector subspace. It is clearly non-empty ($0 \in V$) and for $x, y \in V$ and $k \in \mathbb{R}$:

$$(x + y)(t) = x(t) + y(t) = x(1 - t) + y(1 - t) = (x + y)(1 - t) \text{ and}$$

$$(kx)(t) = kx(t) = kx(1 - t) = (kx)(1 - t),$$

so $x + y \in V$ and $kx \in V$.

Problem 3. Note that $Y + Z$ is nonempty ($0 \in Y + Z$). Let (“fix”) $x_1, x_2 \in Y + Z$ and $k \in K$. Then $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ for some $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ (by definition of $Y + Z$). Then:

$$x_1 + x_2 = y_1 + z_1 + y_2 + z_2 = (y_1 + y_2) + (z_1 + z_2) \in Y + Z \text{ and}$$

$$kx_1 = k(y_1 + z_1) = ky_1 + kz_1 \in Y + Z.$$

Proof that $Y \cap Z$ is a vector subspace left to you.

Problem 4. Review the definition of linear independence and linear dependence!

The vectors $(k, 1, 0)$, $(1, k, 1)$, $(0, 1, k)$ are linearly dependent if there exist scalars α, β, γ , not all equal to 0, such that

$$\alpha(k, 1, 0) + \beta(1, k, 1) + \gamma(0, 1, k) = 0.$$

We rewrite the above condition as:

$$\begin{array}{rcl} k\alpha & +\beta & = 0 \\ \alpha & +k\beta & +\gamma = 0 \\ & \beta & +k\gamma = 0 \end{array}$$

After a Gaussian reduction we arrive at the following equivalent system of equations (hint: to avoid dividing by k which could potentially be 0, start by reordering the equations – moving the first equation to the bottom):

$$\begin{array}{rcl} \alpha & +k\beta & +\gamma = 0 \\ & \beta & +k\gamma = 0 \\ & & k(k^2 - 2)\gamma = 0 \end{array}$$

If $k(k^2 - 2) \neq 0$, then $\gamma = 0$, and it follows that $\beta = 0$ and $\alpha = 0$. Thus, in this case, the given vectors are linearly independent. If $k(k^2 - 2) = 0$, then

the last equation becomes $0 \cdot \gamma = 0$, so γ could be non-zero. So, our system has a non-trivial solution (for example, $\gamma = 1, \beta = -k, \alpha = k^2 - 1$). This means that the given vectors are linearly dependent. If we are working over the field \mathbb{R} , the possible values for k are $0, \sqrt{2}, -\sqrt{2}$. In the case of the field \mathbb{Q} , only $k = 0$ solves the condition $k(k^2 - 2) = 0$.

Remark. For a quick overview of Gaussian reduction see:

<http://www.maths.abdn.ac.uk/igc/tch/eg1006/notes/node130.html>

<http://www.maths.abdn.ac.uk/igc/tch/eg1006/notes/node131.html>

Problem 5 As before, in Problem 2(a), when we want to prove that a general formula is *not* true, we do so by exhibiting a *specific* example for which the formula fails.

Remark. To come up with a counterexample, you would typically draw lots of pictures and try different spaces. The more linear algebra you learn, the more intuition you will have about where to look and what to try. Here are some general hints, which may or may not work:

Hint 1. Start by looking in lower dimensions.

Hint 2. In our ‘everyday’ 3d space, vector (sub)spaces of dimension 1 are the lines through the origin, vector spaces of dimension 2 are the planes through the origin, the vector space of dimension 3 is of course the whole 3d space and the vector space with dimension 0 is $\{0\}$. These are really easy to visualize.

Hint 3. Attempt to prove the claim (eventhough you have been told it is false), and see where the proof fails – this can give you valuable insight for your counterexample!

So here is one example that your classmate came up with:

Working in the vector space \mathbb{R}^2 , let $X = \text{Span}((1, 1))$ (this is the $y = x$ line), $Y = \text{Span}(0, 1)$ (the x -axis) and $Z = \text{Span}((0, 1))$ (the y -axis). Draw the picture (always!) and check that these indeed provide a counterexample!

Problem 6. Review the definition of the linear independence/dependence! If you know it, recite it to yourself, and even write it down. It’s good practice to write down things you know about your problem.

Remark. This was probably not a very difficult problem, but I decided

to write up a full solution, since it is a nice and short illustration of a *proof by contradiction*.

Assume x_1, x_2, \dots, x_n are linearly independent, but suppose, for a contradiction, that there exists i such that $x_i = 0$. But then, the following non-trivial (coefficients not all 0) linear combination equals 0:

$$0 \cdot x_1 + \dots + 0 \cdot x_{i-1} + 1 \cdot x_i + 0 \cdot x_{i+1} + \dots + 0 \cdot x_n = 0.$$

This contradicts that fact that the vectors are linearly independent. So our assumption was false, thus for all i , $x_i \neq 0$.

Problem 7. Remark. Even if not mentioned in the problem, whenever you see, say $\dim(Y)$, it is implicit that dimension of Y is finite. In this problem we can assume that Y and Z are finite dimensional. Then since $Y \cap Z \subset Y$, it follows that $Y \cap Z$ is also finite dimensional. Throughout the proof it will emerge that $Y + Z$ is also finite dimensional. Thus all parts of the equation are well defined. If the problem gives you that the big space X is finite dimensional, then you don't have to worry about anything – you know that all of the subspaces of X are finite dimensional.

Picture! As you read the following paragraph, it's a good idea draw a Venn diagram of the sets Y and Z with non-empty intersection (like, two large intersecting circles) and as we pick points (vectors) in Y and Z enter them in the appropriate place in the diagram. This will help you greatly keep the track of where each vector lives!

$Y \cap Z$ is finite dimensional, say of dimension m , so there exists a basis $\{e_1, \dots, e_m\} \subset Y \cap Z$ for $Y \cap Z$. Now, these vectors are linearly independent in Y , so they can be extended to a basis $\{e_1, \dots, e_m, f_1, \dots, f_p\} \subset Y$ of Y . Similarly, we obtain a basis $\{e_1, \dots, e_m, g_1, \dots, g_q\} \subset Z$ for Z . Notice that $\dim(Y) = m + p$ and $\dim(Z) = m + q$.

Claim. The set $\{e_1, \dots, e_m, f_1, \dots, f_p, g_1, \dots, g_q\}$ is a basis for $Y + Z$. (This is a reasonable guess. For the equation

$$\dim(Y) + \dim(Z) = \dim(Y + Z) + \dim(Y \cap Z)$$

to work out, we need the dimension of $Y + Z$ to be $m + p + q$, and we have exactly that many vectors at our disposal, so why not hope that they are a

basis of $Y + Z$...)

Proof of Claim. We need to prove that the set $\{e_1, \dots, e_m, f_1, \dots, f_p, g_1, \dots, g_q\}$ is linearly independent and spans $Y + Z$. For linear independence, suppose

$$\sum \alpha_i e_i + \sum \beta_j f_j + \sum \gamma_k g_k = 0. \quad (1)$$

Now (here comes a little trick),

$$\sum \alpha_i e_i + \sum \beta_j f_j = - \sum \gamma_k g_k.$$

Call these two equal vectors a vector x , i.e, let

$$x := \sum \alpha_i e_i + \sum \beta_j f_j = - \sum \gamma_k g_k.$$

Now, $x = \sum \alpha_i e_i + \sum \beta_j f_j$, so that $x \in Y$. But also, $x = - \sum \gamma_k g_k$, so $x \in Z$. Thus, $x \in Y \cap Z$. Recall that the basis of $Y \cap Z$ is $\{e_1, \dots, e_m\}$, so x can be written as $x = \sum \delta_l e_l$. Now,

$$x = - \sum \gamma_k g_k = \sum \delta_l e_l,$$

so

$$\sum \delta_l e_l + \sum \gamma_k g_k = 0.$$

This is a linear combination of linearly independent vectors (recall that $\{e_1, \dots, e_m, g_1, \dots, g_q\}$ is a basis of Z), so all the coefficients must be 0. In particular, we get $\gamma_1 = \dots = \gamma_q = 0$. Substituting 0 values for the γ 's in 1, we get

$$\sum \alpha_i e_i + \sum \beta_j f_j = 0.$$

Finally, we notice that this is a linear combination of linearly independent vectors (why?), so $\alpha_1 = \dots = \alpha_m = \beta_1 = \dots = \beta_p = 0$. This completes the proof of linear independence.

To show that $\{e_1, \dots, e_m, f_1, \dots, f_p, g_1, \dots, g_q\} \subset Y + Z$ is a span of $Y + Z$, let $u \in Y + Z$. Then $u = y + z$, for some $y \in Y$ and $z \in Z$. Now $y \in Y$ can be written as the linear combination of e 's and f 's and $z \in Z$ can be written as a linear combination of e 's and g 's. Adding these two up, we get that u can indeed be written as a linear combination of e 's, f 's and g 's, which is what we wanted to prove.