Math 2370 – Fall 2006
Practice Problems VIII
Due October 19 as a HOMEWORK

Problem 1: Let V be a finite dimensional linear space and let $T \in L(V,V)$. Given a subspace W of V, set $T^{-1}(W) = \{x \in V : Tx \in W\}$.
(a) Show that $T^{-1}(W)$ is a subspace of V.
(b) Show that $\dim T^{-1}(W) \le \dim N_T + \dim W$, where N_T is the nullspace of T.
(c) If $S \in L(V,V)$, show that $\text{rank } ST \ge \text{rank } T + \text{rank } S - \dim V$.

Problem 2: An $n \times n$ matrix A is *skew-symmetric* if $A^T = -A$. Let A be a skew-symmetric $n \times n$ matrix with real entries and with n odd.
(a) Show that $\det A = 0$.
(b) Show that all the eigenvalues of A are pure imaginary.

Problem 3: Find all eigenvectors and eigenvalues of the backward shift operator $T \in L(\mathbb{C}^\infty, \mathbb{C}^\infty)$ defined by $T(x_1, x_2, x_3, ...) = (x_2, x_3, ...)$

Problem 4: Suppose that $S, T \in L(V,V)$ where V is finite dimensional
(a) Show that if $\dim R_T = k$, then T has at most $k+1$ distinct eigenvalues.
(b) Show that ST and TS have the same eigenvalues.
(c) Show that if every vector in V is an eigenvector of T then $T = aI$.

Problem 5: Suppose that $T \in L(V,V)$ where V is finite dimensional. Show that there is a basis B in V such that the representation of T with respect to B is an upper triangular matrix. (Do not use the spectral theorem.)

Problem 6: Show that $n \times n$ complex matrix A is never similar to $A + I$.