Math 2370 – Fall 2006
Practice Problems VII

Problem 1: Let A be a block diagonal matrix with the decomposition

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_n \end{bmatrix}$$

where A_i are square. Show that $\det A = \det A_1 \det A_2 \cdots \det A_n$.

Problem 2: Let E be an $n \times n$ matrix with the block decomposition $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A and D are square.

(a) Show that if A is invertible then E can be decomposed as $E = \begin{bmatrix} I & 0 \\ W & I \end{bmatrix}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$.

(b) Using the decomposition show that $\det E = \det A \det(D - CA^{-1}B)$.

Problem 3: (a) Use the adjoint formula to compute the inverse of the following matrix:

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

(b) Use Cramer's rule to solve the following system over the field of rational numbers:

$$3x - 2y = 7$$
$$3y - 2z = 6$$
$$3z - 2x = -1$$

Problem 4: An $n \times n$ complex matrix is *unitary* if $A\overline{A}^T = I$, ($\overline{A}^T$ is the conjugate transpose of A). If A is unitary, show that $|\det A| = 1$.

Problem 5: Show that if the $n \times n$ matrix $(I - AB)$ is invertible then $(I - BA)$ is invertible.

Problem 6: If V is the vector space of matrices over K and B is a fixed $n \times n$ matrix over K, let L_B and R_B be the linear operators on V defined by $L_B(A) = BA$ and $R_B(A) = AB$. Show that $\det L_B = \det R_B = (\det B)^n$.

Problem 7: Let A, B, C, and D be $n \times n$ matrices and E be the $2n \times 2n$ matrix $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Show that $\det E = \det(AD - BC)$ when A, B, C, and D commute.