Math 2370 – Fall 2006
Practice Problems III

Problem 1: Consider the quotient space obtained by reducing the space P of polynomials modulo subspace M of P. If $M = P_m$, the subspace of polynomials of degree less than m, is P/M finite-dimensional ? How about if M is the space of even polynomials ? How about if M is the subspace of polynomials divisible by $p(t) = t^n$?

Problem 2: Show that congruence is an equivalence relation.

Problem 3: Show that if X is the subspace of $\mathbb{C}^{2n}$ consisting of all vectors $(x_1, x_2, ..., x_{2n})$ such that $x_1 = x_2 = ... = x_n = 0$ and Y is the subspace of $\mathbb{C}^{2n}$ of vectors for which $x_j = x_{j+n}$, $j = 1,...,n$ then $\mathbb{C}^{2n} = X \oplus Y$.

Problem 4: If Y and Z are subspaces of a linear space X such that $X = Y \oplus Z$ then Z' is isomorphic to $Y^{\perp}$, Y' is isomorphic to $Z^{\perp}$, and $X' = Y^{\perp} \oplus Z^{\perp}$.

Problem 5: The vectors $x_1 = (1,1,1)$, $x_2 = (1,1,-1)$, and $x_3 = (1,-1,-1)$ form a basis of $\mathbb{C}^3$. If $\{y_1, y_2, y_3\}$ is the dual basis and if $x = (0,1,0)$, find $y_1(x)$, $y_2(x)$, and $y_3(x)$.

Problem 6: Show that if Y and Z are subspaces of a finite-dimensional linear space, then $(Y \cap Z)^{\perp} = Y^{\perp} + Z^{\perp}$ and $(Y + Z)^{\perp} = Y^{\perp} \cap Z^{\perp}$

Problem 7: Define three linear functions on P_3, space of polynomials of degree less than 3, by

$$f_1(p) = \int_0^1 p(t)dt, \qquad f_2(p) = \int_0^2 p(t)dt, \qquad f_3(p) = \int_0^{-1} p(t)dt$$

Show that $\{f_1, f_2, f_3\}$ is a basis for $(P_3)'$ by exhibiting the basis for P_3 of which $\{f_1, f_2, f_3\}$ is the dual basis.

Problem 8: Consider the space $\mathbb{C}^{n\times n}$ of $n \times n$ complex matrices. Show that the trace is a linear function on $\mathbb{C}^{n\times n}$. Find the relation between $\text{trace}(AB)$ and $\text{trace}(BA)$ where $A, B \in \mathbb{C}^{n\times n}$. Use that relation to prove that there are no matrices obeying $AB - BA = I$, where I is the identity matrix.

Problem 9: Let $\mathbb{R}^{n\times n}$ be the linear space of all 2×2 real matrices and let $B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Let Y be the subspace of $\mathbb{R}^{n\times n}$ consisting of all A such that $AB = 0$. Let f be a linear function on $\mathbb{R}^{n\times n}$ that is in $Y^{\perp}$. Suppose that $f(I) = 0$ and $f(C) = 3$. Find $f(B)$.