Math 2370 – Fall 2006
Practice Problems XIII

Problem 1: Which of the following matrices is unitarily similar to a diagonal matrix and why (or why not)?

$$\begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Problem 2: Given the space of P_n complex polynomials of degree less than n in the variable t, with inner product

$$\varphi(f,g) = \int_0^1 f(t)\overline{g(t)}dt$$

(a) Is the multiplication operator T which acts as $Tf(t) = tf(t)$ a self-adjoint map from P_n to itself ?
(b) Is the differentiation operator D a self-adjoint map from P_n to itself?

Problem 3: Let O be 3×3 orthogonal matrix with determinant 1. Show that it represents a rotation about some line in $\mathbb{R}^3$. Find this line and the angle of rotation.

Problem 4: Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find an orthogonal matrix Q such that $Q^T AQ$ is diagonal.

Problem 5: Let V be a vector space endowed with two scalar products $(.,.)_1$ and $(.,.)_2$. Suppose that $(v,v)_1 = (v,v)_2$ for every vector v in V. Then $(v,w)_1 = (v,w)_2$ for every pair of vectors v, w in V.

Problem 6: Let X be a finitely-dimensional Euclidean space and T is a linear map on X. Show that the range of T^* is the orthogonal complement of the nullspace of T.

Problem 7: Let V be the real Euclidean space consisting of real-valued continuous functions on the interval $-2 \le t \le 2$ with the scalar product

$$(f,g) = \int_{-2}^{2} f(t)g(t)dt$$

Let W be the subspace of odd functions. Find the orthogonal complement of W.

Problem 8: Let X be a finitely-dimensional real Euclidean space and let B be a linear map such that $(Bx,x) \ge 0$ for all x in V.
(a) Show that $(Bx,x) = 0$ implies $(Bx,y) + (x,By) = 0$ for all y in V.
(consider the map $t \mapsto (B(x+ty), x+ty), t \in \mathbb{R}$)
(b) Deduce from (a) that the nullspace of B equals the nullspace of B^* and hence that the nullspace and range of B are orthogonal.

Problem 9: Let V be a finite-dimensional real Euclidean space. A linear map T is said to be a reflection with respect to a plane S_u, defined as the span of vectors orthogonal to a given vector u in V, if $T(u) = -u$ and $T(w) = w$ for all w in S_u.

(a) Show that T is given by

$$T(v) = v - 2\frac{(v,u)}{(u,u)}u$$

(b) Show that T is an isometry.

Problem 10: Let V be the space of all $n \times n$ matrices over the reals. For A, B from V define $(A,B) = \mathrm{tr}(B^T A)$.

(a) Show that $(.,.)$ is a scalar product on V

(b) Let E_{ij} be a matrix in V whose i-th row and j-th column entry is 1 and all other entries are 0. Show that the matrices E_{ij} $i, j = 1,2,\ldots,n$ form an orthonormal basis for V.

(c) For A in V let $f(A) = \sum_{i,j=1}^{n}(i+j)a_{ij}$ where a_{ij} $i, j = 1,2,\ldots,n$ are the elements of matrix A. Find a matrix B such that $f(A) = (A,B)$ for all A in V and show that B is unique.