

Math 2370 – Fall 2006
Practice Problems XII

Problem 1: Show that if Y and Z are subspaces of a finite-dimensional Euclidean space then $(Y + Z)^\perp = Y^\perp \cap Z^\perp$.

Problem 2: Show that if A is a linear map of a finite-dimensional Euclidean space into itself then

- (a) $\text{tr}(A^* A) \geq 0$
- (b) $\text{tr}(A^* A) = 0$ if and only if $A = 0$.

Problem 3: Let A be a linear map of a finite-dimensional Euclidean space X into itself.

- (a) Show that $\|Ax\|$ is bounded on the unit sphere $\|x\| = 1$ and attains its maximum there.
- (b) Show that $\|A\| = \max_{\|x\|=\|y\|=1} (Ax, y)$
- (c) Show that $\|A^*\| = \|A\|$

Problem 4: Show that an $n \times n$ real matrix Q is orthogonal if and only if its columns are unit vectors that are pairwise orthogonal.

Problem 5: Let P_3 be the space of real polynomials of degree less than 3. Let $\varphi : P_3 \times P_3 \rightarrow \mathbf{R}$ be the function

$$\varphi(f, g) = \int_0^{\infty} f(t)g(t)e^{-t} dt$$

- (a) Show that φ is a scalar product
- (b) Starting from the basis $(1, t, t^2)$ use Gram-Schmidt algorithm to find an orthogonal basis for P_3 .

Problem 6: Show that if (x_1, x_2, \dots, x_n) is a collection of orthonormal vectors in a Euclidean space X and if x is any vector in X , then $\sum_i |(x, x_i)|^2 \leq \|x\|^2$.

Problem 7: Recall that for real Euclidean space $4(x, y) = \|x + y\|^2 - \|x - y\|^2$. Find an analogous relation expressing (x, y) in terms of the norms of several appropriately chosen linear combinations of x and y that is valid in a complex Euclidean space.

Problem 8: Let X be a real Euclidean space, Y its subspace, and let P be the orthogonal projection of X on Y . Show that $(Px, y) = (x, Py)$ for all x, y in X .

Problem 9: Show that two vectors in a complex Euclidean space are orthogonal if and only if $\|ax + by\|^2 = \|ax\|^2 + \|by\|^2$ for all complex numbers a, b .