Research Statement

Bojana Pejić

December 1, 2006

Introduction

My research has been in topological groups and descriptive set theory, especially in understanding the relationships between the algebraic structure and the topology of a topological group.

Often, the topological structure is fundamentally important for proving results about groups. But conversely, the algebraic structure places very tight restrictions on the types of topology that can be defined on a group. In essence, my work is a part of a wider program to understand what topologies can arise in given algebraic structures.

In particular, my thesis advisor Dr. Paul Gartside and I, have been interested in the extreme case where the collection of group topologies of specified type on a group is small — of cardinality 0 or 1. When does a group admit no more than one locally compact completely metrizable group topology? When does a group admit a unique Polish (i.e. separable completely metrizable) group topology?

These questions form an important part of the greater, and intensively studied, problem of the size and structure of the lattice of group topologies on a topological group. Further, the problems of the existence of a unique topological group topology of some type is closely related to the problem of automatic continuity — under what conditions on groups or homomorphism can we deduce that an algebraic homomorphism between topological groups must automatically be continuous?

Both problems — automatic continuity and uniqueness of certain types of topology — have come to the fore in a diverse range of subject areas. The study of automatic continuity in Banach algebras has been very fertile (see Dales's comprehensive *Banach algebras and automatic continuity*, [1]).

One of the fundamental problems in the theory of profinite (i.e. compact zero-dimensional) groups is *Serre's Conjecture* which essentially asks if every finitely generated profinite group has a unique profinite group topology. And the problem of recovering the model from an automorphism group of a first order structure is equivalent to asking which automorphism groups have a unique Polish group topology.

Research to date

A *Polish group* is a topological group that is separable and metrizable by a complete metric. These groups are ubiquitous in mathematics — to understand a mathematical object one often needs to understand its symmetries, and groups of symmetries of reasonably small objects come naturally equipped with a topological structure making them into a Polish group. Banach spaces, unitary groups of separable Hilbert spaces, autohomeomorphism groups, Lie groups, automorphism groups of first order structures, profinite groups etc. are some of the examples of Polish groups.

Definability The notion of 'definability' of sets plays an important role — proofs of uniqueness and automatic continuity all depend on understanding what sets are 'definable', or 'computable', both algebraically (in terms of the group operation) and topologically. In this context, a set is considered to be 'definable topologically' if it is Borel. By 'algebraically definable', we have in mind such sets as conjugacy classes, commutators, centralizers, powers, etc.

A classic example that illustrates the importance of definability of sets is a theorem of Mackey [4]: A Polish group G has a unique Polish group topology if there is a countable point-separating family of sets that are Borel in *any* Polish group topology on G. The difficulty in applying this result is deciding which sets are Borel in *any* Polish group topology. The ideal candidates for the point-separating collection would be algebraically definable sets that we know must be Borel.

Definable sets come from free words, w. In any topological group topology 'identity sets', $w(\cdot, \ldots)^{-1}(1)$, are necessarily closed. For example, a centralizer of and element is an identity set, and so must be closed. Centralizers (and minor variations) are what Kallman [3] used to show that the automorphism group of a manifold has a unique Polish group topology. Using identity sets, we have shown that the infinite symmetric group S_∞ (the group of permutations on countably many elements) has a unique Polish group topology.

'Verbal sets', $\{w(g_1, \ldots, g_n; \ldots) \mid g_1, \ldots, g_n \in G\}$, on the other hand, are analytic (continuous images of Borel sets), but it is not clear if they are Borel. Examples of verbal sets include the set of the n-th powers, the conjugacy classes, commutators, and so on. It is unfortunate that verbal sets are analytic, while Mackey's theorem requires Borel sets. An important problem, then, is which verbal sets are Borel. From the theory of Polish group actions, conjugacy classes are indeed Borel. We investigated the question if, in fact, all verbal sets are Borel. We proved that in S_∞, all verbal sets of the form $\{w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G\}$ are Borel. We also proved that the set of squares in the group $Homeo(I)$ of autohomeomorphisms of the unit interval is Borel. However, the following result that we have obtained dispels the hope that all verbal sets might be Borel:

Theorem 1 *In the group $Homeo(S^1)$ of the autohomeomorphisms of the unit circle, the set of squares is completely analytic, and hence* not *Borel.*

Though the statement of this result is simple, its proof is difficult and uses interesting techniques from descriptive set theory. Also, this result is somewhat surprising, in contrast to the situation in $Homeo(I)$ — identifying the endpoints of the unit interval to give the circle results in squares becoming much more complex! These results can be found in our paper [2], currently submitted for publication. The article is available at the URL http://www.math.pitt.edu/ boka/research.html.

Alternatively, instead of working on showing which verbal sets are Borel, one can work to strengthen Mackey's theorem to apply to analytic sets. Mackey's result follows from: if X is a Polish space and $\mathcal{A}$ a countable, point-separating family of Borel sets in X, then the σ-algebra of Borel sets in X is generated by $\mathcal{A}$. Is it in fact the case that if $\mathcal{A}$ is a countable, point-separating family of *analytic* sets in X, then the σ-algebra of Borel sets in X is contained in the σ-algebra generated by $\mathcal{A}$? (This would suffice to prove Mackey theorem for analytic sets.) We suspect not. Indeed we have shown this to be false for sets just a little further up the projective hierarchy:

Proposition 2 *Let $\mathcal{C} = 2^{\mathbb{N}}$ be the Cantor space. There is a countable point separating family $\mathcal{A}$ of sets in $\mathcal{C}$, each of which is the union of an analytic and a co-analytic set, such that $\sigma(\mathcal{A})$ does not contain $Borel(\mathcal{C})$.*

While we do not think that a countable point separating family of analytic sets guarantees the uniqueness of the Polish group topology, we have recently obtained a necessary condition for the uniqueness of Polish group topology that works with analytic sets:

Theorem 3 *If G is a Polish group with a neighborhood base at the identity of sets that are analytic in any Polish group topology on G, then G has a unique Polish group topology.*

In particular, since verbal sets are always analytic, we have that:

Theorem 4 *If G is a Polish group with a neighborhood base at 1_G of identity and verbal sets, then G has a unique Polish group topology.*

Using this result with verbal sets, we were able to establish the uniqueness of Polish group topology for profinite groups and compact Lie groups:

Theorem 5 *If G is a finitely generated profinite group, then G has a unique Polish group topology.*

Theorem 6 *If G is a compact, connected, simple Lie group with trivial center (for example, $SO(3,\mathbb{R})$), then G has a unique Polish group topology.*

We remark here that since no countable family of identity sets in compact, connected Lie groups separates point, Mackey's Theorem with identity sets could not have been used to give us Theorem 6.

Proposed research

We have seen that identity and verbal sets are immensely useful, but they do have limitations (identity sets may not be sufficient and not all verbal sets are Borel). So far, our applications have involved constructing concrete collections of identity and verbal sets. We would like to move beyond the identity and verbal sets. In future, I would like to investigate more powerful methods that would allow us to assert the existence of the sets we need, without having to explicitly construct them.

Turning to more general questions, consider the following statements about a topological group G, and a property $\mathcal{P}$ of topological groups:

(Aut) Every (abstract group) automorphism of G is continuous.

(U$\mathcal{P}$) The group G has a **unique** group topology satisfying $\mathcal{P}$.

(N$\mathcal{P}$) The group Ghas **no** group topology satisfying $\mathcal{P}$.

(AC$\mathcal{P}$) Every homomorphism $\phi : G \to H$, where H is a topological group satisfying $\mathcal{P}$, is continuous.

Specific instances of properties $\mathcal{P}$ that we are interested in include: 'the topology is compact','the topology is completely metrizable','the topology is Polish', 'the topology is locally compact separable metrizable', 'the topology is profinite'.

The following implications are easy to check for a topological group G with property $\mathcal{P}$:

i) (U$\mathcal{P}$) iff every isomorphism $\phi : G \to H$, where H is a topological group satisfying $\mathcal{P}$, is continuous.

ii) Provided $\mathcal{P}$ is finitely productive, (AC$\mathcal{P}$) iff every monomorphism $\phi : G \to H$, where H is a topological group satisfying $\mathcal{P}$, is continuous.

iii) (AC$\mathcal{P}$) $\implies$ (U$\mathcal{P}$) $\implies$ (Aut).

General Question 1 *For each of the properties $\mathcal{P}$ of interest: which, if any, of the implications in iii) are reversible?*

General Question 2 *For any pair $\mathcal{P}, \mathcal{Q}$ of properties of interest: does (U$\mathcal{P}$) imply (U$\mathcal{Q}$)? does (AC$\mathcal{P}$) imply (AC$\mathcal{Q}$)?*

References

[1] H. G. Dales, *Banach algebras and automatic continuity*, Oxford University Press, New York, 2001.

[2] P. Gartside and B. Pejić, *The complexity of the set of squares in the homeomorphism group of the unit circle*, Submitted for publication. `http://www.math.pitt.edu/~boka/research.html` (2006).

[3] Robert R. Kallman, *Uniqueness results for homeomorphism groups*, Transactions of the American Mathematical Society **295** (1986), 389–396.

[4] G. W. Mackey, *Borel structures in groups and their duals*, Transactions of the American Mathematical Society **85** (1957), 134–165.